

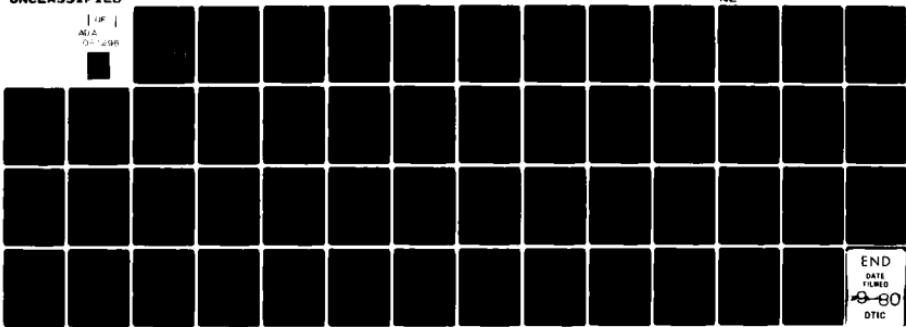
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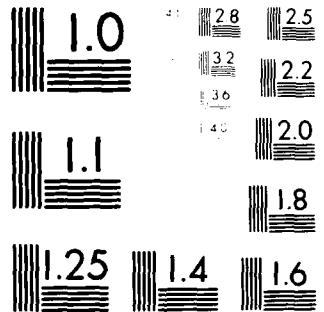
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THE PRESSURE FOOTPRINT OF A TURBULENT LAYER
 ON A COMPLIANT BOUNDARY

Submitted to

U. S. Office of Naval Research
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Abstract. We examine one way in which a compliant boundary may affect the low wave number pressure field measured at the surface under a turbulent boundary layer. Specifically, we assume that the low wavenumber pressure field measured at the wall results from the large scale motions in the outer part of the flow, and that these are triggered by the bursting in the buffer layer, which is caused in its turn by secondary instabilities growing on inflectionary profiles caused by longitudinal vortices in the sub- and buffer layers. We examine the direct effect of the compliant boundary on the growth of these longitudinal vortices, using the energy method. We represent the compliant boundary by a half space filled with a linear visco-elastic medium with a single time constant. We find that the effect of the boundary is to introduce an effective slip in the boundary condition on the disturbance (the longitudinal vortices), producing a boundary conditions intermediate between the inviscid boundary condition and the no-slip boundary condition. Since this boundary condition is less dissipative, the vortices are less stable, and have a higher growth rate. They might consequently be expected to grow faster and occur more frequently, if this were the only mechanism involved, resulting in a shift of the low wave-number pressure spectrum to higher frequencies. This appears to be the opposite of observations, suggesting that other mechanisms are involved.

1. Scientific background. As early as Lumley (1964) the possibility was considered that a compliant surface might alter the structure of a turbulent boundary layer. The interest at that time was in drag reduction; it was hoped that a compliant surface might change the stability characteristics of the large eddies in the viscous sublayer and buffer region, resulting in a thicker sublayer, corresponding to a reduction of drag. A preliminary analysis was carried out, which indicated that realistic compliant surfaces were probably always detrimental. The analysis, however, was limited to progressive waves; a treatment of the more realistic longitudinal vortices had been possible since Serrin (1959), but Lumley was unaware of this work until 1967 (see Lumley, 1971). At the time (1964) the matter was not pursued, since other forms of drag reduction appeared more interesting. The subject has recently been reexamined (Bushnell et al, 1977), and considerable experimental work exists (some of it indicating reduction of drag); the theoretical picture is not clear, although a modification of the large eddy flow preceding a burst is thought to be implicated.

A possibility that was overlooked at the time, but that may be of interest now, is that alteration of the structure of the turbulent boundary layer by a compliant surface might alter the pressure field observed at the surface, particularly in the low wavenumber region, i.e., the region of disturbances having scales of many boundary layer thicknesses. This would permit the modification, by the application of a compliant treatment to a rigid surface, of the low wavenumber pressure field perceived at the rigid surface, something not otherwise possible without structural modification, since the application of a relatively thin compliant surface treatment would not influence the transmission of pressure fields in this wavenumber range.

Examination of such a possibility can more easily be undertaken now for several reasons: first, the results of Serrin (1959), and in particular Lumley (1971), already referred to, which make possible the treatment of the longitudinal vortices; second, an understanding of the dynamical role of these vortices, arising in part from studies of polymer drag reduction (Lumley, 1977); and finally, the existence of much recent data on wall pressure fluctuations and their relation to boundary layer structure (see Willmarth & Bogar, 1977).

The mechanism we have in mind, to connect the large eddies in the sub- and buffer layers with the pressure distribution on the wall is this (see Willmarth & Bogar, 1977): the large eddies are in the form of counter-rotating streamwise vortices; in the updraft between two vortices, the streamwise velocity profile develops an inflection point, due to the slow moving fluid from the vicinity of the wall which is lifted up into the higher velocity region; the inviscid instability resulting from this inflectionary profile produces a burst of Reynolds stress; this burst of Reynolds stress results in a temporary change in the thickness of the boundary layer, associated with a fluctuation in the pressure which is felt throughout the thickness of the layer, and in particular at the wall.

Hence, the key to the whole mechanism is the frequency of bursting and the growth rate of the large eddies in the sub- and buffer layers. Existing data is reasonably consistent regarding the scaling of the size of these large eddies, which are observed to scale with the wall variables. Unfortunately, the scaling of the frequency of bursting is less secure: there is data to indicate scaling both with wall variables and with outer layer variables. It is thus not at all clear what mechanism controls the bursting frequency. We will examine here the possibility that the growth rate of the eddies controls the bursting frequency; that is, that the life-time of the eddy is inversely proportional to the growth rate during the time of most rapid growth; that the eddies are amplified from a primordial soup of disturbances in the sublayer (containing all possibilities) and that consequently a new eddy will appear as soon as possible after the last one has died; hence, that the frequency of appearance will be controlled by the life-time.

In order to determine the growth rate of the eddies, we will use the energy method (Lumley, 1971). In this method, we ask for the form of the disturbance which has the largest growth rate at a given instant, subject to given boundary conditions, and for a given mean velocity profile. Such an optimal disturbance will not remain optimal as it evolves, and was not optimal before the instant of analysis. Among other effects, the presence of the disturbance will progressively erode the mean velocity profile (due to the Reynolds stress produced by the disturbance), and the reduced strain rate of the mean profile will reduce the energy supply of the disturbance, slowing its growth rate, and increasing its size. We will not analyze this

non-linear effect here, which has not yet been analyzed for the rigid wall, but will content ourselves with an analysis of the influence of the boundary condition corresponding to a compliant surface on the growth rate of these optimum disturbances, so that we can have a direct comparison with Lumley (1971).

2. Extremization with variable viscosity. We represent the increased momentum transport of the turbulence in the boundary layer by a variable viscosity; this has known drawbacks (Tennekes & Lumley, 1972), but Lumley (1971) shows that very little of the turbulent momentum transport enters into the dynamics. Specifically, the dynamics are dominated by the region very near the wall, where the strain rate and the momentum transport are large, and the latter is predominantly viscous.

We begin from the Navier-Stokes equations written with variable viscosity, for the undisturbed motion U_i and for the combination of undisturbed motion plus disturbance, u_i . Multiplying by the disturbance velocity and summing, an equation for the disturbance energy $e = u_i u_i / 2$ can be written. This equation is now integrated over the region in which the flow is taking place; we assume that the undisturbed boundaries are not moving (at least not normal to themselves). Finally, we introduce the notation

$$\sigma \int e dV = (d/dt) \int e dV \quad (2.1)$$

Using the divergence theorem to rewrite all transport terms as surface integrals, we finally have

$$\begin{aligned} \sigma \int e dV &= - \int S_{ij} u_i u_j dV - 2 \int v_s s_{ij} dV \\ &\quad - \oint [u_j e - u_i \tau_{ij} / \rho] d\sigma_j \end{aligned} \quad (2.2)$$

where S_{ij} is the mean strain rate $(U_{i,j} + U_{j,i})/2$ and s_{ij} the strain rate of the disturbance motion $(u_{i,j} + u_{j,i})/2$. $\tau_{ij} = -p\delta_{ij} + 2\mu s_{ij}$ is the stress tensor of the disturbance motion.

The volume integral may be over the entire region in which the flow is taking place. However, if the region is unbounded in one or more directions, we will suppose that the motion is periodic in that direction, and we will take the integral only over a period.

The first term in the surface integral represents the transport of disturbance energy across the boundary by the disturbance velocity. The contribution to the integral from the parts of the surface that lie in the flow, in the case of a periodic motion, and that are consequently at the beginning and end of a period, will cancel, so that we need concern ourselves only with that part of the integral which lies on an actual bounding surface. We can eliminate this term using either of two arguments. First, we suppose that the surface is nearly rigid, so that the motion of the surface is relatively small. Essentially, we can imagine expanding in a parameter related to the surface rigidity. Then the disturbance velocity at the surface is a small quantity, which vanishes when the surface is rigid. The surface stress is of order unity, since it does not vanish as the surface becomes rigid. Consequently, the second term in the surface integral is of first order, while the first term is of third order, and can be neglected by comparison.

A second argument, which is unrelated to the amplitude of the surface motions, supposes (as we will suppose) that the disturbance is periodic cross-stream. Then the disturbance energy, being quadratic, will contain components at zero wavenumber and at twice the wavenumber of the basic disturbance; consequently, the energy flux which appears in the surface integral will contain components at the basic wavenumber and at three times the wavenumber, all of which will integrate to zero over one or more periods of the basic disturbance.

We must now extremize equation (2.2) (suppressing the energy flux term in the surface integral). We wish to consider only incompressible motions, so we must add a Lagrange multiplier to the equation to maintain incompressibility, which will take the form of a pressure. We add to the right

hand side the following:

$$0 = 2 \int (\phi/\rho) \delta u_{i,i} dV = 2\phi(\phi/\rho) \delta u_i d\sigma_i - 2 \int (\phi_{,i}/\rho) \delta u_i dV \quad (2.3)$$

where ϕ is the Lagrange multiplier. This gives for the extremized form

$$\begin{aligned} & \int (\sigma u_i + 2s_{ij} u_j - 4(v_s)_{,j} + 2\phi_{,i}/\rho) \delta u_i dV \\ &= \phi(\delta u_i \tau_{ij}/\rho + u_i \delta \tau_{ij}/\rho - 2\tau_{ij}^\phi \delta u_i/\rho) d\sigma_j \end{aligned} \quad (2.4)$$

where

$$\tau_{ij}^\phi = -\phi \delta_{ij} + 2\mu s_{ij} \quad (2.5)$$

We may presume that the variation in the disturbance velocity in the interior of the fluid, and at the surface, are independent (since the variations at any two points are independent); hence, the volume integral gives (under the usual assumption that the integrand is continuous)

$$\sigma u_i/2 + s_{ij} u_j = -\phi_{,i}/\rho + 2(v_s)_{,j} \quad (2.6)$$

We cannot remove the integrand from the surface integral yet, because we have not related the variation in the surface stress to the variation in the disturbance velocity.

3. The Viscoelastic Substrate. We begin by assuming that the wall is a semi-infinite region, a half-space. There are, of course, other possibilities: we could consider a visco-elastic slab of finite thickness over a rigid surface, or other wall compositions with complex compliance varying with depth in different ways. The principal way in which such walls would differ from the half-space we have chosen, would be in the existence of reflection from the compliance changes. Non-dimensionalized with sublayer variables, the time required for a disturbance to travel through the substrate a distance equal to the sublayer thickness is about 11.6, if the wall material is of the same density as the fluid, and Young's modulus is equal to the dynamic head based on the shear velocity. Non-dimensionalized in the same way, the time required for substantial growth of the eddy is roughly 4, using values corresponding to a rigid wall. This is, of course, a very compliant wall; as the wall is stiffened, the transmission time will be reduced (proportional to the inverse of the square root of Young's modulus). At the same time, as the eddy grows, and the shear is reduced, the growth rate will decrease, increasing the growth time. Hence, the times are probably comparable, with the reflection time likely being the shorter for stiffer walls. Thus, there is a possibility that reflection plays a role, and this bears looking into. Here, however, we concentrate on the mechanisms present without reflection. We will find that the principal mechanism has to do with the displacement of the surface, which requires a compensating disturbance velocity to cancel the undisturbed velocity at the new location of the surface (see section 4.)

If we write s_i for the displacement vector at each point in the substrate, the equations of motion may be written as

$$\ddot{s}_i = -p_{,i}/\rho' + \tau_{ij,j}^v/\rho' \quad (3.1)$$

Primed parameters refer to wall materials. The stress tensor τ_{ij}^v in the substrate is given by

$$\tau_{ij}^v = G(s_{i,j} + s_{j,i})/2 + \mu'(d/dt)(s_{i,j} + s_{j,i})/2 \quad (3.2)$$

In our extremization procedure we say nothing about the temporal evolution of the disturbance. The quantity σ which appears in the equations is the instantaneous relative growth rate of the integrated energy. There is no implication that the growth rate is the same at different places in the fluid, or will be the same at other instants before or after the instant for which the equations are written. In fact, Lumley (1971) shows that the solution to the energy method equations necessarily has a non-linear evolution, with different growth rates at different points, and that it is optimum only at the instant of analysis, evolving through non-optimum states before and after, and hence necessarily having different growth rates. In order to obtain the stresses in the substrate, however, we must have a value for the relative growth rate of the displacement, and for its second derivative. In the absence of other information, the simplest choice appears to be to suppose that the displacements in the substrate are growing exponentially in a neighborhood of the instant of analysis, and that the relative growth rate is one half of the relative growth rate of the integrated disturbance energy (the same as the growth rate of the disturbance velocity, if it were everywhere the same). Hence, we write

$$\dot{s}_i = s_i \sigma/2, \ddot{s}_i = s_i (\sigma/2)^2 \quad (3.3)$$

which permits us to write

$$\tau_{ij}^v = (G + \mu' \sigma/2)(s_{i,j} + s_{j,i})/2 \quad (3.4)$$

As a disturbance in the fluid, we will be interested in a velocity field that is not a function of the streamwise coordinate, and is periodic cross-stream, because this is the type of disturbance which is observed in the sublayer over a rigid wall (see Lumley, 1971) as well as over a compliant one (see Hansen et al, 1980). Consequently, we will be interested in disturbances of the substrate of the same type, and we will assume

$$\begin{aligned} s_2 &= \psi_{,3}, & s_3 &= -\psi_{,2}, & ()_{,1} &= 0 \\ \psi &= \hat{\psi} \exp[ikx_3], & s_1 &= \hat{s} \exp[ikx_3] \end{aligned} \tag{3.5}$$

Taking account of the radiation condition (requiring that the disturbances in the substrate vanish at infinity), the solutions to the equations of motion may be written as (in terms of arbitrary boundary conditions)

$$\begin{aligned} \hat{\psi} &= (\lambda \hat{\psi}_0 - \hat{\psi}'_0/k) \exp[kx_2]/(\lambda - 1) + (\hat{\psi}'_0/k - \hat{\psi}_0) \exp[k\lambda x_2]/(\lambda - 1) \\ \hat{s} &= \hat{s}_0 \exp[k\lambda x_2] \end{aligned} \tag{3.6}$$

where

$$\lambda = [1 + \sigma^2 \rho' / 2k^2 (G + \mu' \sigma / 2)]^{1/2} \tag{3.7}$$

The stress tensor at the surface may now be written in terms of the arbitrary boundary conditions, where we obtain the pressure from the cross stream component of (3.1):

$$-\hat{p}_0 = i(G + \mu' \sigma/2)(\lambda \hat{\Psi}_0 - \hat{\Psi}'_0/k)k^2(1 + \lambda)/2 \quad (3.8)$$

so that the symmetric stress is given by $\tau_{ij} = -\dot{\hat{p}}_0 \delta_{ij} + \tau_{ij}^v =$

$$\left\{ \begin{array}{l} i(1 + \lambda)(\lambda \hat{\Psi}_0 - \hat{\Psi}'_0/k)k^2/2 \\ \lambda \hat{s}_k/2 + i\lambda(1 + \lambda)\hat{\Psi}'_0 k^2/2 + i(1 - \lambda)\hat{\Psi}'_0 k/2 \\ i \hat{s}_k/2 \quad \{(\lambda - 1)\hat{\Psi}_0 - (1 + \lambda)\hat{\Psi}'_0/k\}k^2/2 \quad i\lambda(1 + \lambda)\hat{\Psi}'_0 k^2/2 - i(3 + \lambda)\hat{\Psi}'_0 k/2 \end{array} \right. \quad \left. \right\}$$

$$\times (G + \mu' \sigma/2) \quad (3.9)$$

Now, we are interested in the surface response, so that we may write on the surface

$$\tau_{ij} d\sigma_j = g_{ik} \hat{s}_k d\sigma \quad (3.10)$$

which becomes, on the bottom surface

$$\tau_{i2} = g_{ik} \hat{s}_k \sigma/2 \quad (3.11)$$

From (3.9) and the definition (3.7), then, we may write

$$g_{ik} = \left\{ \begin{array}{ccc} \lambda/(\lambda^2 - 1) & 0 & 0 \\ 0 & \lambda/(\lambda - 1) & i/(\lambda + 1) \\ 0 & -i/(\lambda + 1) & 1/(\lambda - 1) \end{array} \right\} \rho' \sigma/2k \quad (3.12)$$

4. The boundary condition. Since the surface of the viscoelastic substrate moves, it interacts with the fluid motion other than simply through the stress on the surface. In particular, suppose that the surface moves normal to itself into the fluid, without moving laterally. The no slip condition must still be maintained; the undisturbed velocity at the new location of the surface is now non-zero and positive, so that a negative streamwise disturbance velocity at this location is required to cancel the former.

The classical kinematic boundary condition takes a somewhat different form here; the part of the condition relating to the slope of the surface is second order, and may be neglected; at the same time, we have a new condition resulting from the possibility of lateral motion. The general condition becomes:

$$\begin{aligned}\dot{s}_i &= u_i(x_1 + s_1, s_2, s_3) = \delta_{1i} U(s_2) + u_i(x_1 + s_1, s_2, s_3) \\ &= \delta_{1i} U' s_2 + u_i^0 + u_{i,j}^0 s_j + \dots\end{aligned}\tag{4.1}$$

Now, if the wall were rigid, the disturbance velocities at the surface would vanish, as would their gradients in the plane of the surface; hence, most of the third and subsequent terms in (4.1) will be of second or higher order, and may be neglected if the surface displacements are small. The gradients normal to the wall of the disturbance velocities parallel to the wall do not vanish as the wall displacement vanishes, however, and hence may not be taken to be small. To avoid the introduction of non-linearities (which do not seem to be worth the additional effort they would require since we expect these results to be of at most qualitative significance), we will assume that the disturbance shear is small relative to the mean shear. Then the relation between the surface displacements and the disturbance velocities at the surface becomes:

$$\begin{aligned}\dot{s}_1 \sigma/2 &= \dot{u}_2^0 2U'/\sigma + \dot{u}_1^0 \\ \dot{s}_2 \sigma/2 &= \dot{u}_2^0 \\ \dot{s}_3 \sigma/2 &= \dot{u}_3^0\end{aligned}\tag{4.2}$$

Note that the lateral surface displacement may not be in the expected direction, corresponding to the value of the lateral disturbance velocity in the fluid at the surface, if the vertical disturbance velocity (combined with the undisturbed shear and the growth rate) has the right value. We will find, in fact, that this is a dominant effect.

Now, on the bottom surface we may write

$$\tau_{i2} = g_{i1} (\hat{u}_2^0 U' / \sigma + \hat{u}_1^0) + g_{i2} \hat{u}_2^0 + g_{i3} \hat{u}_3^0 = \bar{g}_{ij} \hat{u}_j^0 \quad (4.3)$$

We may now return to the surface integral part of (2.4). If we suppose (corresponding to the type of disturbance in which we are interested) that

$$\tau_{i2} = \tau_{i2}^* \exp[ikx_3] \quad u_i = \hat{u}_i \exp[ikx_3] \quad (4.4)$$

the surface integral consists simply in an integral in the cross stream direction, which may be carried out, giving

$$\begin{aligned} \tau_{i2}^* \delta\hat{u}_i^* / \rho + \tau_{i2}^* \delta\hat{u}_i^* / \rho + \hat{u}_i^* g_{ij}^* \delta\hat{u}_j^* / \rho + \hat{u}_i^* g_{ij}^* \delta\hat{u}_j^* / \rho \\ - 2\tau_{i2}^* \delta\hat{u}_i^* / \rho - 2\tau_{i2}^* \delta\hat{u}_i^* / \rho = 0 \end{aligned} \quad (4.5)$$

If (4.3) is substituted, we have

$$\tau_{i2}^* = \hat{u}_j (\bar{g}_{ij} + \bar{g}_{ji}^*) / 2 \quad (4.6)$$

Use of (3.12) and (4.3) permits evaluation of the coefficient matrix to give

$$(\tilde{g}_{ij} + \tilde{g}_{ji})/2 = \begin{Bmatrix} \lambda/(\lambda^2 - 1) & \lambda U'/\sigma(\lambda^2 - 1) & 0 \\ \lambda U'/\sigma(\lambda^2 - 1) & \lambda/(\lambda - 1) & i/(\lambda + 1) \\ 0 & -i/(\lambda + 1) & 1/(\lambda - 1) \end{Bmatrix} \rho' \sigma/2k \quad (4.7)$$

5. Solution of the stability problem. We suppose a disturbance of the form

$$\begin{aligned} u_1 &= \tilde{u} \exp[ikx_3] \\ u_2 &= \psi_{,3} = ik\tilde{\psi} \exp[ikx_3] \\ u_3 &= -\psi_{,2} = -\tilde{\psi}' \exp[ikx_3] \end{aligned} \quad (5.1)$$

We may obtain an expression for the pressure at the surface from the lateral component of equation (2.6), giving

$$-\phi^0 = [-\tilde{\psi}' \sigma/2 + v_m(D^2 - k^2)\tilde{\psi}']\rho/ik \quad (5.2)$$

where a subscript zero indicates evaluations at the wall.

Using this, (5.1) and (4.7), we may evaluate the stress in the fluid, and obtain the boundary conditions explicitly as

$$\begin{aligned} \mu \tilde{u}' &= [\tilde{u}' \lambda/(\lambda^2 - 1) + ik\tilde{\psi}' U' \lambda/\sigma(\lambda^2 - 1)]\rho' \sigma/2k \\ [-\tilde{\psi}' \sigma/2 + v_m(D^2 - 3k^2)\tilde{\psi}']\rho/ik &= [\tilde{u}' \lambda U'/\sigma(\lambda^2 - 1) + ik\tilde{\psi}' \lambda/(\lambda - 1) \\ &\quad - i\tilde{\psi}' /(\lambda + 1)]\rho' \sigma/2k \end{aligned} \quad (5.3)$$

$$-\mu(D^2 + k^2)\tilde{\psi}' = [k\tilde{\psi}' /(\lambda + 1) - \tilde{\psi}' /(\lambda - 1)]\rho' \sigma/2k$$

We wish now to introduce the same non-dimensionalization as used in Lumley (1971); that is, we non-dimensionalize lengths by the thickness of the viscous sublayer (defined as the point of intersection between the sublayer profile and the logarithmic profile), and introduce as a small parameter ϵ the inverse of the Reynolds number defined using this length, the shear velocity and the wall viscosity. The dimensionless growth rate

is defined as $\beta^2 = \sigma/U'$. Indicating dimensionless variables by the same letters, the boundary conditions become

$$\begin{aligned} 2k\epsilon^2 \hat{u}' &= \beta^2 (\hat{u}' + ik\psi'/\beta^2) \lambda \rho' / \rho (\lambda^2 - 1) \\ \epsilon^2 (D^2 - 3k^2) \psi' &= \beta^2 [i\lambda \hat{u}' / \beta^2 (\lambda^2 - 1) - k\psi' \lambda / (\lambda - 1) \\ &\quad + \psi'' [1 + (\lambda + 1)\rho/\rho'] / (\lambda + 1)] \rho' / 2\rho \end{aligned} \quad (5.4)$$
$$-2\epsilon^2 (D^2 + k^2) \psi' = \beta^2 \{k\psi' / (\lambda + 1) - \psi'' / (\lambda - 1)\} \rho' / \rho 2k$$

The expression for λ may be written as

$$\lambda^2 = 1 + \beta^4 \rho' / \rho 2k^2 \epsilon^2 (G/\rho u_*^2 + \beta^2 \rho' v' / \rho v_m) \quad (5.5)$$

where primed parameters correspond to wall material. We have in mind constructing an asymptotic expansion for large Reynolds number, so that the region of viscous influence on the disturbance motion will become thin and can be handled by matching, as in Lumley (1971). There are a number of ways that the compliant wall can be incorporated in this scheme, corresponding to different limiting processes. Since in any event we are interested primarily in learning the qualitative effects of the compliance, we would be satisfied by some sort of expansion about the rigid wall case. The easiest way to accomplish this is to take the Young's modulus proportional to the square of the Reynolds number, so that the wall becomes more and more rigid as the Reynolds number increases. This has the effect of holding λ constant as the Reynolds number goes to infinity. Then if we write

$$\begin{aligned} \psi &= v_0 (g_0 + \epsilon^2 g_1 + \dots) \\ \hat{u} &= v_0 (h_0 + \epsilon^2 h_1 + \dots) \end{aligned} \quad (5.6)$$

we find that the lowest order solution corresponds to the rigid wall case, with $h_0 = g_0 = g_0' = 0$ at the wall. To second order we obtain

$$\begin{aligned}
 \beta^2(h_1^0 + g_1^0 ik/\beta^2) \lambda \rho' / \rho (\lambda^2 - 1) &= 2k h_0^0 = D i \kappa^2 / 2 \beta \\
 \beta^2 [i h_1^0 \lambda / \beta^2 (\lambda^2 - 1) - k g_1^0 \lambda / (\lambda - 1) + g_1^0 [1 + (\lambda + 1) \rho / \rho']] & \\
 / (\lambda + 1)] \rho' / 2 \rho & \quad (5.7) \\
 \beta^2 [k g_1^0 / (\lambda + 1) - g_1^0 / (\lambda - 1)] \rho' / \rho 2 k &= -2 g_0^0 = 2 D \beta^2
 \end{aligned}$$

The expressions for the lowest order solutions will now be the same as in the rigid wall case, and may be taken directly from Lumley (1971). These have been used to give the final expressions in (5.7), where D is a negative constant which must be determined by matching.

We may solve (5.7) to obtain

$$\begin{aligned}
 h_1^0 &= -2iD(\lambda - 1) \{1 + k^2(\lambda + 1)/4\beta^2 - 2k(\lambda - 1)[1 + (\lambda + 1)\rho/\rho']/\beta(\lambda + 1)\}\rho \\
 &/\rho' \beta \{\lambda[1/\beta^4(\lambda + 1) - 1] + (\lambda - 1)^2[1 + (\lambda + 1)\rho/\rho']/(\lambda + 1)^2\} \quad (5.8)
 \end{aligned}$$

$$\begin{aligned}
 g_1^0 &= D(\lambda - 1) \{2\beta + k^2/2\beta^5 + 4k(\lambda - 1)[1 + (\lambda + 1)\rho/\rho']/(\lambda + 1)\}\rho \\
 &/\rho' k \{\lambda/\beta^4(\lambda + 1) - \lambda + (\lambda - 1)^2[1 + (\lambda + 1)\rho/\rho']/(\lambda + 1)^2\} \quad (5.9)
 \end{aligned}$$

$$\begin{aligned}
 g_1^{0'} &= -D(\lambda - 1) \{4k - (\lambda - 1)\{2\beta + k^2/2\beta^5 + 4k(\lambda - 1) \\
 &[1 + (\lambda + 1)\rho/\rho']/(\lambda + 1)\} / (\lambda + 1)\{\lambda/\beta^4(\lambda + 1) - \lambda \\
 &+ (\lambda - 1)^2[1 + (\lambda + 1)\rho/\rho']/(\lambda + 1)^2\}\} \rho / \rho' \quad (5.10)
 \end{aligned}$$

In Lumley (1971) it is shown that, in order to have one turning point in the outer solution, so that the eddies close toward the outside of the layer (rather than producing upward and downward jets which persist to the outside of the layer), it is necessary to have $1/2\beta^4 > 2$. With this restriction, it is clear from (5.8-10) that $g_1^{0'}$ is positive real, g_1^0 is negative real and h_1^0 is positive imaginary. (We presume that the outer stream function is positive real, corresponding to D negative). In figure 1 we indicate the inviscid (outer) stream function and its first derivatives, together with the amplitude of the streamwise disturbance; we have shown dotted the modification due to viscosity over a rigid wall, and the dashed curve indicates the modification due to compliance. It is seen that the compliant solution is intermediate between the inviscid and the viscous solution. We would thus expect the compliant solution to be less

dissipative (more like the inviscid solution) and hence more unstable. Note that, although the lateral velocity at the wall is in the direction that might be expected based on the outer solution, both the normal velocity and the streamwise velocity are reversed. This may be attributed directly to the effect of the boundary condition in section 4.

Now, we must obtain the solution for the g_1 corresponding to the boundary conditions (5.8-10) (since we will do the matching to obtain the stability criterion entirely in terms of g_1 , we will not need h_1). Since the rigid-wall boundary conditions on g_1 vanish, it is convenient to write g_1 as the sum of the rigid wall solution and a particular solution corresponding to the non-zero boundary conditions, which is easily shown to be

$$g_1 = \dots + g_1^0 + n g_1^{0'} \quad (5.11)$$

(where the ellipsis indicates the rigid-wall solution). The matching may now be carried out exactly as in Lumley (1971), resulting in an additional term on the left of the $O(\epsilon^3)$:

$$-B + g_1^0 = \dots \quad (5.12)$$

(the ellipsis indicating the unaltered right hand side of the corresponding equation in Lumley, 1971) and on the left of the $O(\epsilon^2\delta)$:

$$\beta[B + (9/4 + 4\beta^4)k^2D/24\beta^6] + g_1^{0'} = \dots \quad (5.13)$$

Adding these expressions results in one additional term on the right hand side of w_3 :

$$w_3 = \dots - k(1/4\beta^4 - 1)^{1/2}(g_1^0 + g_1^{0'}/\beta)/DB \quad (5.14)$$

Finally, the eigenvalue condition is obtained:

$$3\pi/4 - Y_0 k = -(1/4\beta^4 - 1)^{1/2} k\epsilon/\beta - (1/4\beta^4 - 1)^{1/2} (1/48\beta^4 + 1) k^3 \epsilon^3 / 2\beta^3 - (1/4\beta^4 - 1)^{1/2} (g_1^0 + g_1^{0'}/\beta) \epsilon^3 k / D\beta \quad (5.15)$$

This condition is the same as in Lumley (1971) except for the additional term on the right hand side.

For a fixed value of the growth rate, the condition (5.15) gives in general two values of wavenumber for each value of Reynolds number ($1/\epsilon$); there is a minimum value of Reynolds number, corresponding to a critical value of wavenumber. A typical curve is shown in figure 2. In Lumley (1971) this minimum value was found by differentiating (5.15) with respect to wavenumber, and requiring that both the value and the derivative vanish. Here, however, the situation is algebraically so complex (due to the appearance of wavenumber in the expressions for surface response) that it was felt to be simpler to determine the minimum value of Reynolds number numerically from (5.15) for each fixed value of growth rate. The resulting plot is shown in figure 3, where the rigid wall case is also shown. It is seen that the growth rate curve is shifted upward, so that a larger growth rate is experienced for a given Reynolds number with a compliant surface than with a rigid surface, as was anticipated from the modification of the boundary condition, which was seen to be less dissipative.

6. Discussion. Lumley (1971) is included as an appendix to this report for convenience. Comparison of the rigid wall curve in figure 3 with the corresponding curve in Lumley (1971) indicates a considerable difference, which has been traced to a numerical error in the computations made for the curve in Lumley (1971). The equations in Lumley (1971) produce the curve in figure 3, which is correct. The curve in figure 3 may be rationalized physically in the following way: note that the wavenumber is roughly the same as that for the Couette flow (the left-hand curve in Lumley 1971, which is correct); hence, under the same shear, the eddies are about the same size, whether an upper wall is present or not. The Reynolds number, on the other hand, is considerably increased, though not doubled; this is

because the outer half of the eddy is unsupported energetically - the shear in this region is very much reduced, so that this part of the eddy must be supported dynamically by the part near the wall.

Note that the tendency of the curves in figure 3 to bend back to the right as they approach the abscissa is not real, but results from the failure of the various approximations (high wave number, high growth rate) that were made in the course of the calculation. The curves should be extrapolated in the same way as were the curves in Lumley (1971).

The lambda indicated on figure 3 is the same as that used in Lumley (1971), and is not the lambda of this report. The lambda of figure 3 is the growth rate non-dimensionalized by sublayer variables.

The very small difference of figure 3 between the rigid and compliant wall (despite the very compliant values of the parameters chosen) is disappointing, as is the fact that it is in the wrong direction. We must conclude that substantial interaction with a compliant surface does not exist with eddies of this form. Probably only for a progressive wave motion does a strong interaction exist. The fact that the change is in the wrong direction results from the fact that the primary effect of the compliant surface here is to allow an apparent slip at the wall, due to the interaction of the surface motion and the undisturbed shear. If we are to have a favorable change, we must seek another mechanism.

Several possibilities come to mind. First is the possibility that the compliant surface interferes with some other phase of the Reynolds stress production mechanism (say, for example, with the secondary instability resulting from the inflectionary profile); this would allow the eddies to grow larger before bursting, increasing the time between bursts, resulting in a shift of the pressure spectrum toward lower frequencies.

It is also possible that some as-yet-undetermined mechanism is responsible for the initiation of the large eddies (and hence for their frequency of occurrence) and that the compliant surface interferes with this in some way. Some light may be shed on this question by experiments on the influence of polymer additives on the time between bursts.

It is also conceivable that the finite thickness of the compliant surface is significant, since the time for transmission and reflection of a signal through the layer is of the order of the growth period; the reflected disturbance could interact favorably with the eddy at a later phase in

its life.

Finally, eddies such as we envision also arise due to the nonlinear interaction of two oblique progressive waves. It may be that such progressive waves are present in the sublayer (a proper orthogonal decomposition theorem applied to the turbulent field gives an Orr-Sommerfeld equation, which has such solutions; see Lumley 1967). Such waves would be much more strongly affected by a compliant boundary.

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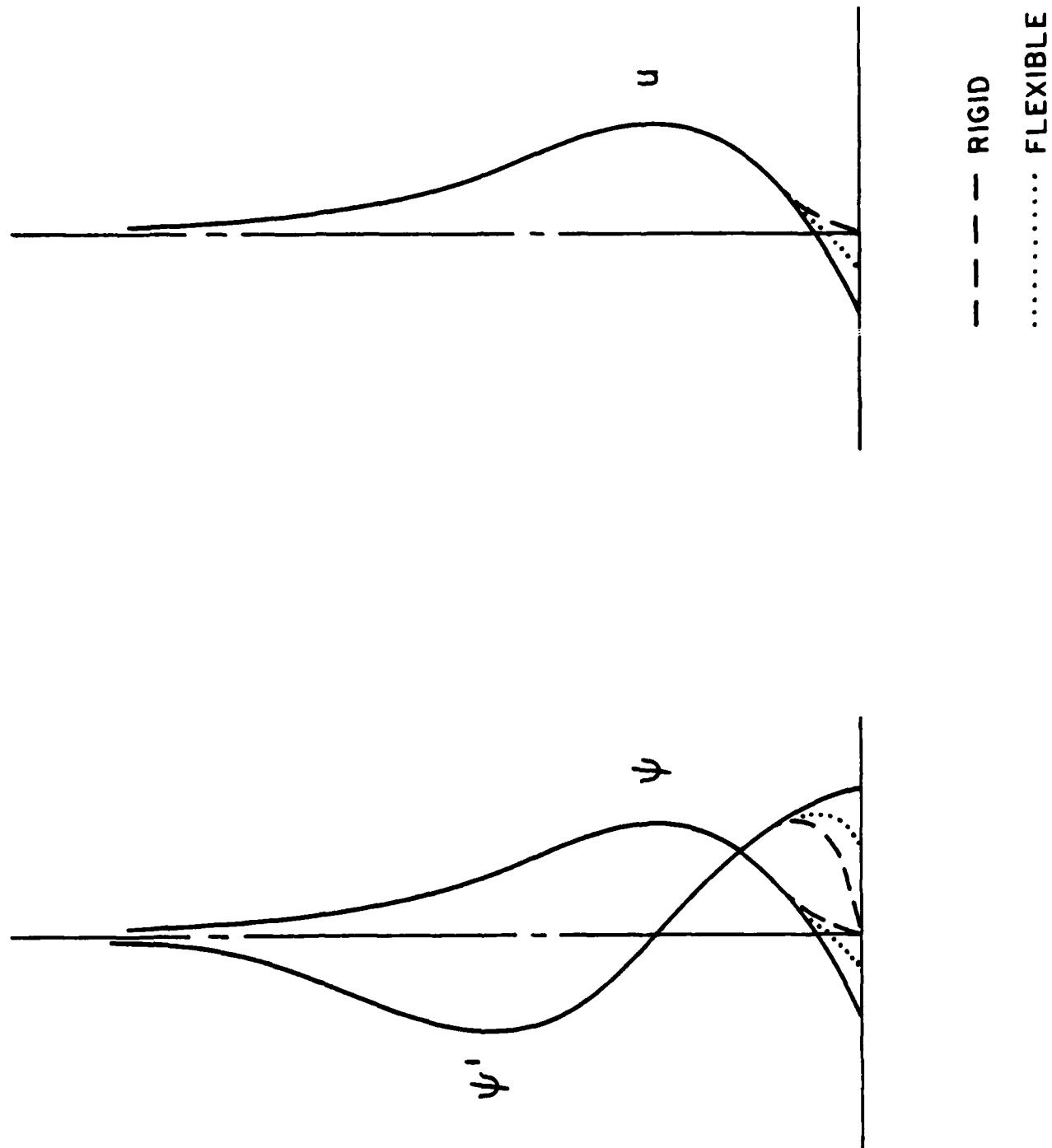


Figure 1. Sketch of the complex modulus of the stream function for the cross-stream velocity perturbation and the streamwise velocity perturbation for both the rigid wall and flexible wall cases. The solid curves correspond to the inviscid (outer) solution. It can be seen that the compliant wall results in boundary conditions intermediate between the rigid and inviscid cases.

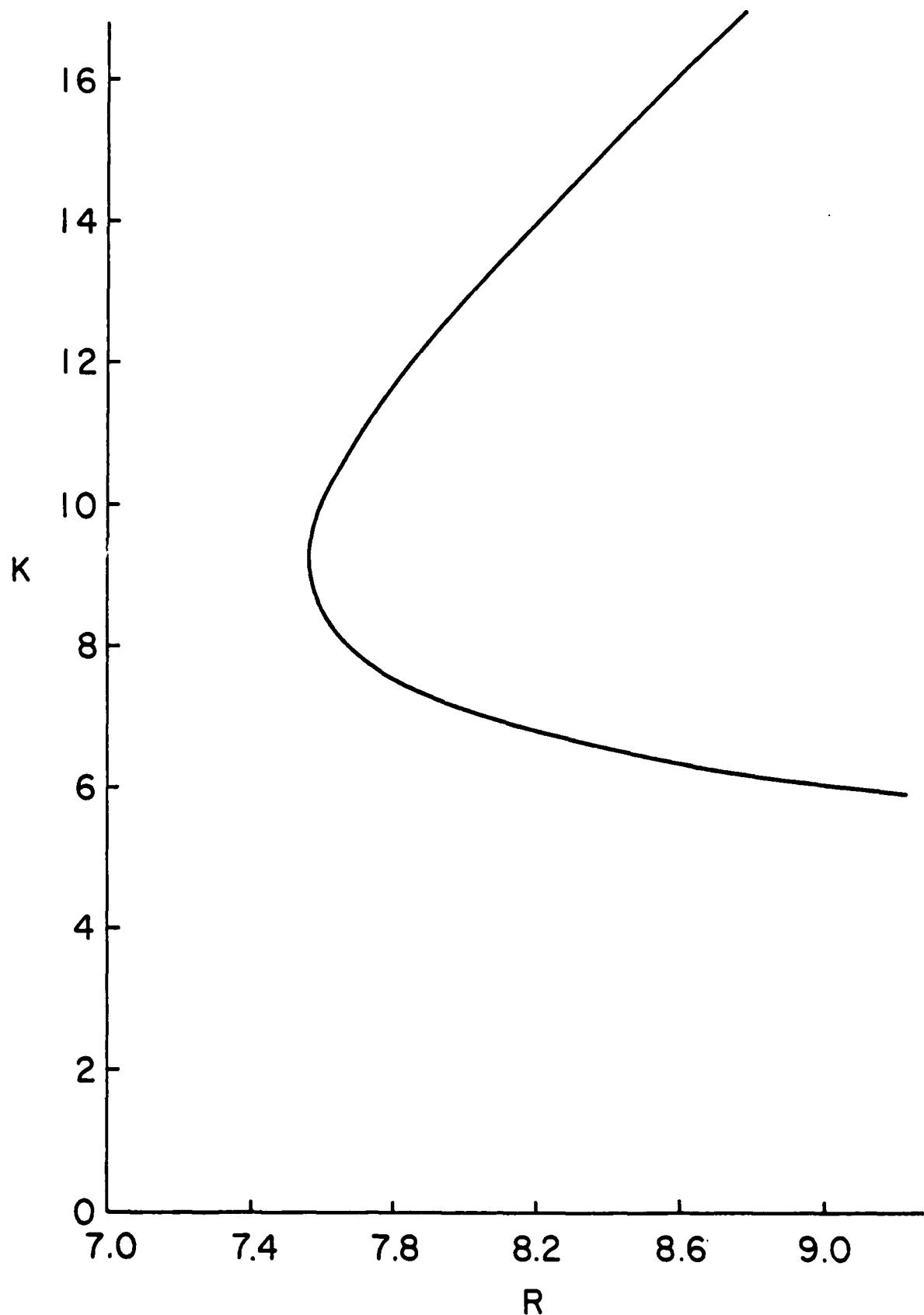


Figure 2. Plot of equation 5.15 for a fixed value of normalized growth rate for the rigid wall case (the compliant wall case is virtually indistinguishable).

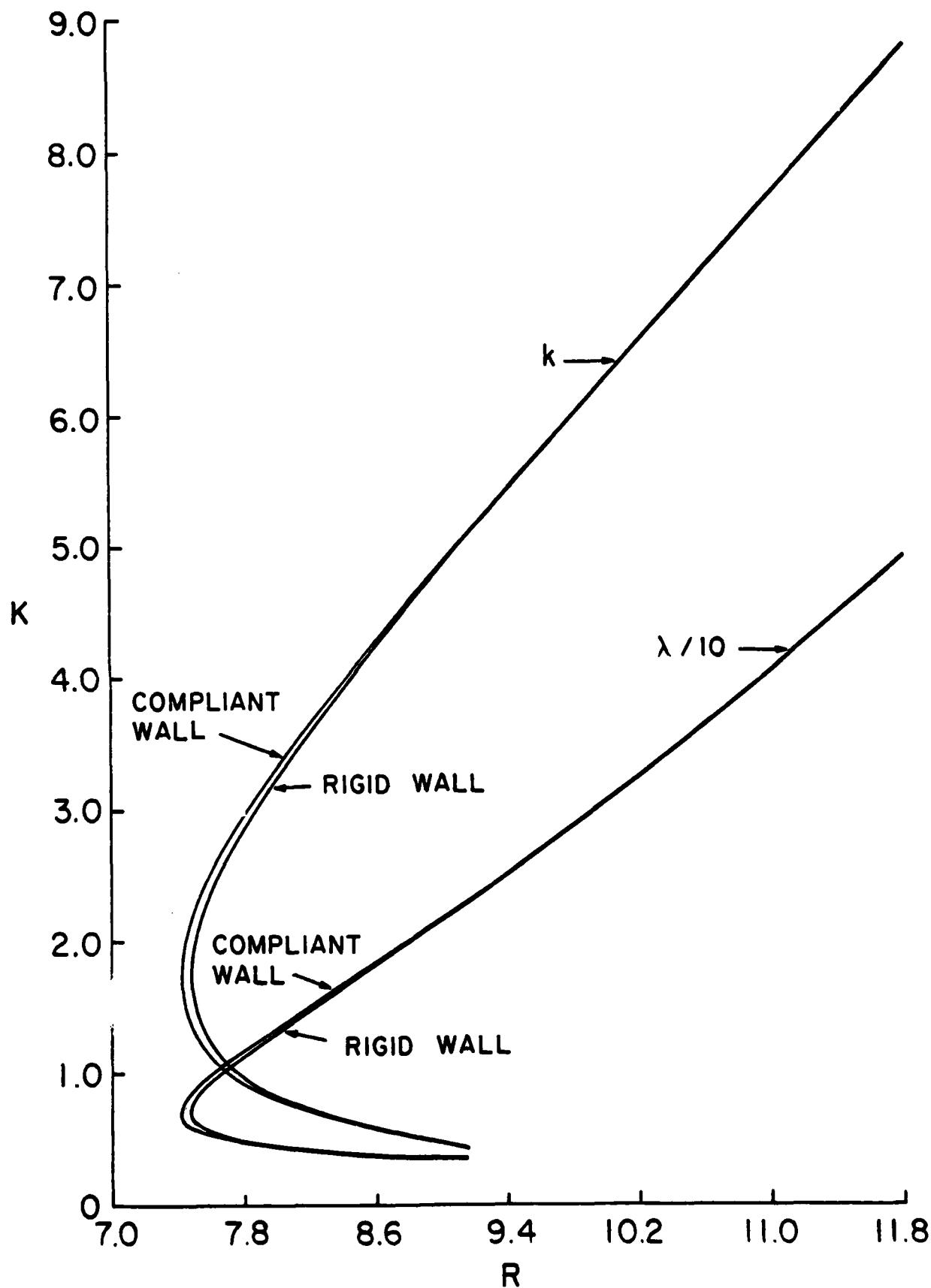


Figure 3. Plot of critical growth rate and wavenumber versus Reynolds number for the rigid wall and compliant wall cases. These are the wavenumbers corresponding to the minimum value of Reynolds numbers for each growth rate from curves such as that shown in Figure 2. Note that lambda here is as defined in Lumley (1971), that is, the growth rate normalized by sublayer variables.

Some Comments on the Energy Method

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ABSTRACT

It is suggested that the large organized motions which are observed in inhomogeneous turbulent flows may be those the net energy of which has the largest growth rate (the smaller scale turbulence being replaced by a constitutive relation). This is similar to the classical energy method for stability analysis; a modern extension of this method (by Serrin, 1959) is described, and the implications of the method carefully evaluated. It is concluded that the method describes dynamically possible motions; the work of Petrov (1938), in which the contrary conclusion is reached, is examined. The motions are found to be necessarily nonlinear, and to evolve. The method is applied to longitudinal rolls in the wall region of boundary layer flow (found by Joseph, 1966, to be least stable in Couette flow, and observed in the wall region). The resulting equations are solved by asymptotic techniques; these make clear that unstable eddies experience viscosity only near the wall; that streamwise disturbances are produced from the mean gradient by vertical disturbances; that in the viscous region transverse disturbances decay, while still producing streamwise disturbances. As a check on the asymptotic analysis, it is also applied to rectilinear Couette flow, where it gives the classical value for the zero-growth-rate Reynolds number (Serrin, 1959; Joseph, 1966). The eigenvalue relation for the wall layer indicates very large growth rates (in sublayer variables) at observed Reynolds numbers, in agreement with observation. Predicted wavenumbers are larger than observed; a way is suggested by which the reduction in shear due to the presence of the eddies would reduce the wavenumber to that observed.

INTRODUCTION

It is now well known that inhomogeneous turbulent flows exhibit recurrent structures in the velocity field, which are referred to as "large eddies" (Townsend, 1956). The dynamical role which these motions play was discussed at length in Townsend (1956), and several guesses made as to the form of eddy required to explain the measurements in various flows. Suggestions have been made (Lum-

ley, 1967, 1970b) as to how these motions might be identified objectively in correlation data. The dynamical equations obeyed by these eddies have been discussed (Townsend, 1956; Lumley, 1967). Both discussions suggest replacing the turbulent velocities of smaller scale by a constitutive relation; that is, treating the large eddy deterministically, as weather is treated meteorologically, while the smaller scale turbulence is treated statistically. Although this is justified in meteorology, it is probably not in flows in the laboratory; it is not hard to show that a spectral gap half a decade wide is required, for such a treatment to be rigorously justified (Lumley, 1970c). In laboratory flows, there is sometimes a gap of sorts, but it is neither wide enough nor deep enough. Nevertheless, there are even less propitious applications of this technique (to cases with no gap at all) which have been remarkably successful, such as the Heisenberg (1948) treatment of the spectrum. Certainly this approach does not violate physical laws (such as the second law of thermodynamics), it can be made to meet global requirements (such as dissipating the proper amount of energy) and can be endowed with certain known characteristics of the real situation (such as viscoelastic behavior; c. f. Lumley, 1970a); hence there is a certain justification in hoping for qualitatively correct results. The method, in addition has the virtue of simplicity.

The equations of motion for the large eddy are now reduced to the Navier-Stokes equations, if a simple eddy viscosity is assumed, or some more complicated equivalent, if a more realistic constitutive equation is adopted. Although this represents a considerable simplification, it still does not make possible straight forward calculation: the number of known exact (or even asymptotic) solutions to the Navier-Stokes equations is remarkably small. Ordinarily, for given boundary conditions, the variety of solutions that can be generated by various initial conditions is considerable; which one is the large eddy? Accordingly, another principle appears to be needed, to isolate the large eddy. It has been suggested (Lumley, 1967) that the mean velocity profile might be neutrally stable (in the small disturbance sense) to the large eddy, thus identifying the latter as the eigenfunction of the Orr-Sommerfeld (or equivalent) equation corresponding to the minimum value of Reynolds number. Unfortunately, direct calculation (Reynolds, Tiederman, 1967) indicates that turbulent boundary layers, at any rate, are stable to all such disturbances.

We are proposing here another principle, that the large eddy observed is that motion which can most efficiently extract energy from the mean motion and give up as little as possible to the (turbulent) dissipation. This is the motion which can therefore grow largest under given circumstances. Application of this principle gives a well defined and relatively simple eigenvalue problem, at least for the case of a Newtonian constitutive relation. In this case, it is formally equivalent to what is known classically as the energy method of stability analysis.

We will restrict ourselves to a Newtonian constitutive relation, since we have in mind applying this analysis to the wall region of a boundary layer or pipe flow. In such a flow, the maximum shear and minimum viscosity is in the sublayer, which is Newtonian; one may reasonably expect that the dynamics of such large eddies as exist may be dominated by Newtonian phenomena. In fact, we will find from the asymptotic analysis that the nature of the effective viscosity outside the sublayer has no influence on the energy balance of the eddies. This would certainly not be generally true; in attempting to calculate the form of large eddies in shear layers, for example, one might expect the character of the effective constitutive relation to be important, since the maximum shear region occurs where the turbulence is strong; the energy balance then will be between production and turbulent dissipation, and the effective constitutive relation is certainly not Newtonian (Lumley, 1970a).

THE ENERGY METHOD OF SERRIN

The classical energy method is associated with Orr (1907). This was limited to two-dimensional disturbances to the basic flow, and continuity was satisfied by the use of a stream function. In 1959, Serrin extended the method to three-

dimensional disturbances, using Lagrange multipliers. Our derivation will be restricted to a Newtonian constitutive relation, and is a trivial generalization of Serrin's, using a slightly different method.

We are seeking the disturbance u_i to the basic flow U_i which, subject to the condition

$$u_{i,i} = 0 \quad (1)$$

has, at a fixed time t , for a fixed value and distribution of v , an extremal value of

$$\sigma = (1/2E)dE/dt \quad (2)$$

where E is the global disturbance energy. That is, the disturbance which can most efficiently extract energy from the basic flow, losing as little as possible, will have the largest growth rate. That this extremism is truly a maximum can be seen more mathematically by considering that, for a fixed distribution of v , the dissipation (which must always be nonnegative) may be made arbitrarily large by selecting u_i having sufficiently small characteristic length scales. The production is bounded (relative to the energy) above and below (by the negative of the least and greatest eigenvalues of the strain rate tensor of the basic flow). Due to the dissipation, however, the range of growth rates is bounded only above, and unbounded below.

Using the integral of the equations of motion, we have from (2)

$$\sigma = [-\int S_{ij} u_i u_j dV - \int u_{i,j} v^2 s_{ij} dV] / \int u_i u_i dV \quad (3)$$

where S_{ij} and s_{ij} are respectively the strain rate tensors of the basic flow and of the disturbance. Note that all transport terms vanish, since they are conservative. Applying the usual techniques of the calculus of variations, we obtain (indicating by σ^+ the maximum value)

$$\int [\sigma^+ u_i + S_{ij} u_j - 2(v s_{ij})_{,j}] \delta u_i dV = 0 \quad (4)$$

Now, if the variation is over incompressible motions, the δu_i are not independent, but are related by

$$\delta u_{i,i} = 0 \quad (5)$$

We may specify δu_1 and δu_2 independently, for example, and then δu_3 is given. If we write

$$\sigma^+ u_i + S_{ij} u_j - 2(v s_{ij})_{,j} = X_i \quad (6)$$

for convenience, then (4) becomes

$$\int X_i \delta u_i dV = \int (X_1 \delta u_1 + X_2 \delta u_2 + X_3 \delta u_3) dV = 0 \quad (7)$$

If we set $X_3 = -\pi_{,3}/\rho$, where π is an arbitrary function, then integration by parts (using the condition that u_i and δu_i either vanish on the boundaries or satisfy a cyclic boundary condition) gives

$$\begin{aligned} \int X_i \delta u_i dV &= \int (X_1 \delta u_1 + X_2 \delta u_2 + \delta u_{3,3} \pi/\rho) dV \\ &= \int (X_1 \delta u_1 + X_2 \delta u_2 - [\delta u_{1,1} + \delta u_{2,2}] \pi/\rho) dV \quad (8) \\ &= \int [X_1 + \pi_{,1}/\rho] \delta u_1 + (X_2 + \pi_{,2}/\rho) \delta u_2 dV \end{aligned}$$

Since δu_1 and δu_2 are independent, we have

$$\sigma^+ u_i + S_{ij} u_j = -\pi_{,i}/\rho + 2(v s_{ij})_{,j} u_{i,i} = 0 \quad (9)$$

This is the same as that derived in Serrin (1959) with the exception of the inclusion of a non-zero growth rate, and variable viscosity. The pressure π is that required to assure (1), and hence may be regarded as a Lagrange multiplier.

This has been applied by Serrin (1959) to Couette flow with uniform viscosity and $\sigma^+ = 0$. Two-dimensional disturbances of the same flow were, of course, calculated by Orr (1907); he found that the Reynolds number at which $\sigma^+ = 0$ (the critical Reynolds number) for such disturbances is 6.65, based on the shear velocity and the half width. Although Orr (1907) said "Analogy with other problems leads us to assume that disturbances in two dimensions will be less stable than those in three; . . ." as pointed out by Joseph (1966), Serrin (1959) found that longitudinal rolls could survive to a lower Reynolds number, namely 4.54 (on the same basis); evidently such disturbances can extract energy more efficiently. Serrin (1959) did not show that this type of disturbance was the most efficient, i. e. - that it corresponded to the largest growth rate ($=0$) at a given Reynolds number, or to the lowest Reynolds number at a given growth rate. Joseph (1966) however, by considering the combined problem of Couette flow of a Boussinesq fluid heated from below, showed that, in fact, the longitudinal rolls are the most efficient.

Taylor (1960) suggested using this type of analysis for a purpose different from ours: namely, viewing a Couette flow as two viscous sublayers face to face, the thickness Reynolds number (defined as above) corresponding to a maximum growth rate of zero appears to give the thickness of the layer next to the wall in which no disturbance can survive without importing energy from above. He used the analysis of Lorentz (1907) who did not actually solve the extremum problem, but guessed at a solution. Using Serrin and Joseph's solution, this would suggest a viscous sublayer thickness of $y^+ = 4.54$, which is near the point at which the mean velocity profile bends away from the linear, although well below the point (~ 9) at which measured dissipation first exceeds production (Townsend, 1956).

Before applying (9) to a specific calculation, we must examine the implications of the method, so that we will know what to expect of it.

CRITIQUE OF THE ENERGY METHOD

The energy method, as applied to flows without buoyancy*, has been subject to extensive criticism. For example, Lin (1955) states: "Even here only the lower limit for the critical Reynolds number can be expected, because stability must be established for all disturbances in the present method, while in reality only those satisfying the hydrodynamic equations need to be considered. The inclusion of the spurious disturbances . . ." (page 59) and "The energy equation thus gives critical Reynolds numbers that are too low," (page 61). Serrin (1959) states: "It is important to note that the energy method cannot provide accurate knowledge of the limits of stability, such as can be gained from the linearized perturbation theory . . . The reason is that in the energy method one establishes stability relative to arbitrary disturbances, while in reality only those satisfying the hydrodynamical equations need be considered." (page 4). Joseph (1966) states "The functions which are admitted for review need not be possible solutions of the conservation equations (momentum and energy). It is in this sense that one may speak of dynamically inadmissible disturbances." (page 181). And further: "Evidently . . . the energy method reflects a sensitivity to spurious and dynamically inadmissible disturbances." (page 182). None of these allegations is supported. However, Monin and Yaglom (1971) state "Analyzing this fact [that the energy method gives lower critical Reynolds numbers than the linear theory] Petrov (1938) came to the conclusion that the values of τ at which

*with $\sigma^+ = 0$, and uniform viscosity.

$F[\psi]$ takes a maximum, taking into account the time variation of all the functions, will apparently in no case generate a dynamically possible motion. Thus the energy method can never give an exact value of $Re_{cr} \min$, but is only suitable for making preliminary, very rough estimates of this value."

The question of why the critical Reynolds numbers predicted by the energy method in flows without buoyancy are substantially lower than those predicted by the linear theory* is certainly an interesting one, although whether they are "too low" as stated by Lin (supra) and in what sense, remains to be seen. The explanations given by Lin, Serrin and Joseph are clearly not satisfactory, since functions are "admitted for review" at a fixed time only; any velocity field at a fixed time satisfies the hydrodynamical equations, which serve to determine its evolution, i.e. - the time derivative. The only substantial argument appears to be that of Petrov (1938).

Since the paper of Petrov (1938) appears never to have been translated, a translation of the relevant section (pages 20-24) is appended (some obvious misprints in the equations have been corrected). The remainder of the Petrov paper is a concise introduction to classical small disturbance stability theory and the energy method, and need not concern us here. Petrov, of course, is reasoning primarily about linearized disturbances having exponential behavior in time.

His argument may be summarized as follows: the small disturbance equations obtained from the Navier-Stokes equations, and the equations obtained from the energy method are not the same; both cannot be satisfied simultaneously (the difference is Petrov's equation (22)); hence, disturbances considered in the energy method do not satisfy the (small disturbance limit of the) Navier-Stokes equations.

This conclusion is correct when applied to the small disturbance limit of the Navier-Stokes equations (though misleading); it is not correct for the full equations. The difference lies in the interpretation of the time derivative. The energy method specifies a global growth rate; locally the growth rate may be greater or less, with the non-linear terms transferring energy from one region to another as required. These non-linear transport terms, being conservative, do not appear in the integrals, so that the equations of the energy method apply to large disturbances whether this is intended or not. The difference between the equations of the energy method and the Navier-Stokes equations serves to determine the local growth rate. Only in the linearized case, where no mechanism is present to transport energy from one level to another (in a parallel flow) is the growth rate required to be the same at all levels, and only then does inconsistency result. Hence, it is the assumption of linearity that is inconsistent. Let us examine the equations.

The condition (2) does not imply that $\partial u_i / \partial t = c u_i$ at each point in the fluid at the instant in question. The growth rate of the disturbance may be larger or smaller than optimum locally, so long as (2) is satisfied. In fact, of course, u_i must also satisfy the Navier-Stokes equation

$$\dot{u}_i + U_{i,j} u_j + u_{i,j} U_j + u_{i,j} u_{j,i} = - p_{,i} / \rho + (\nu(u_{i,j} + u_{j,i}))_{,j} \quad (10)$$

and the local growth rate can be obtained by subtracting (9) from (10)

$$\dot{u}_i - \sigma^+ u_i = - \Omega_{ij} u_j - u_{i,j} U_j - u_{i,j} u_{j,i} - (p - \pi)_{,i} / \rho \quad (11)$$

* In Joseph (1966) it is shown that in several flows with buoyancy, the critical Reynolds numbers predicted by the energy method are close to, or the same as, those predicted by the linear theory.

where $U_{i,j} = S_{ij} + \Omega_{ij}$, Ω_{ij} being the antisymmetric part. As required by (2), multiplication of (11) by u_i and integration over the region gives

$$dE/dt = (d/dt) \int u_i u_i dV/2 = \int u_i u_i dV = \sigma + \int u_i u_i dV \quad (12)$$

If our explanation is correct, equation (11), when restricted to a linearized two-dimensional disturbance of a two-dimensional parallel flow, should give Petrov's equation (22). Let $U_i = (U(x_2), 0, 0)$,

$$\begin{matrix} 0 & U'/2 & 0 \\ S_{ij} = & U'/2 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 0 & U'/2 & 0 \\ \Omega_{ij} = & -U'/2 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad (13)$$

where $U' = U_{x_2}$. Let us satisfy (1) by the use of a stream function $\psi_1 = \psi_{,2}, \psi_2 = -\psi_{,1}$. Eliminating the pressure between 1 and 2 components of the linearized (11) gives

$$(\partial/\partial t)v^2\psi - \sigma^+ v^2\psi = U''\psi_{,1}/2 - UV^2\psi_{,1} - U'\psi_{,21} \quad (14)$$

Now, in the linear case (considering constant, uniform viscosity) the coefficients of (10), or of the equation for the stream function obtained from it, are independent of x_1, t , and hence the solution may be written as a sum of terms of the form

$$\psi = \phi e^{ia(x_1 - ct)} \quad c = c_r + ic_i \quad (15)$$

where, from (2) and (15) we necessarily have

$$\sigma^+ = ac_i \quad (16)$$

Substituting this in (14) we obtain immediately Petrov's equation (22) -

$$(U - c_r)(\phi'' - a^2\phi) - U''\phi/2 + U'\phi' = 0 \quad (17)$$

That this is inconsistent is clear because, as Petrov points out, one parameter (the viscosity) is absent from (17); hence an eigen-solution of (17) cannot be consistent with an eigen-solution of the Orr-Sommerfeld equation (obtained from the linearized equation for the stream function obtained from (10), on substitution of (15)), since the latter contains the viscosity.

Hence, we must conclude that the extremal disturbance generated by the energy method is necessarily non-linear, entailing transport of disturbance energy, and necessarily has a local growth rate of the disturbance energy differing from the global growth rate, but averaging spacially to the latter. The disturbance does satisfy the hydrodynamical equations, which serve to determine the local growth rate. Since the class of disturbances considered is no smaller than, and may be larger than that considered in the linear theory (which necessarily have no transport, and the same growth rate everywhere) the Reynolds number obtained for the extremal value $\sigma^+ = 0$ is no larger than, and may be less than, that obtained from the linear theory.

As an aside, it may be noted that, since the (necessarily) non-linear disturbance generated by the energy method for $\sigma = \sigma^+$ (necessarily) has local growth rates different from σ^+ it will evolve in form; at an instant following (or preceding) the instant of analysis, it will have changed and may no longer be optimum. But, if the optimum disturbance evolved continuously through optimum states, equation (9) would have to be satisfied in some neighborhood. We would thus have two eigenvalue problems in this neighborhood: either (9) and (10), or equivalently (9) and (11). But we may use Petrov's argument that the sets (9) and (11) cannot be simultaneously satisfied, due to the absence of the viscosity

in (11). Hence, (9) and (11) can at most be simultaneously valid at a point, and the optimum disturbance must evolve to one which is not optimum, and hence has a growth rate less than the extremal value σ^+ . Hence, the disturbance energy has, at the instant of analysis, a global growth rate of $2\sigma^+$, and in some neighborhood of that time a smaller growth rate. At no time can the energy have a larger growth rate than the optimal one; if the optimal disturbance were not unique, the disturbance energy might at later times evolve to other optimum states, producing points of growth rate equal to $2\sigma^+$; by the preceding argument, however, these must be isolated, so that the disturbance energy has a growth rate almost everywhere less than $2\sigma^+$. Hence, the optimal disturbance energy is bounded from above by the exponential with growth rate $2\sigma^+$ (a result of Serrin, 1959, obtained in a different way), and has this growth rate at the instant of analysis; at other times, the growth rate is almost everywhere smaller. Thus, optimum disturbances obtained for $\sigma^+ = 0$ have a disturbance energy which is monotone decreasing almost everywhere.

We must note that we have not excluded the possibility that disturbances which have, at the instant of analysis, positive global growth rates, will ultimately decay. These would also be counted stable. For this reason also (and perhaps primarily) the critical Reynolds number given by the energy method for $\sigma^+ = 0$ is conservative; it assures that no disturbance can grow ultimately, and that at a higher value some disturbance can grow initially; it does not guarantee that there is a disturbance which can grow ultimately at a higher Reynolds number. Thus, one says that at a given Reynolds number the energy method determines stability, but cannot determine instability. In fact, Joseph has recently proved (1971) that longitudinal rolls in a shear flow, although they have an initially increasing disturbance energy, always decay ultimately.

We may speculate that, in flows with heat transfer, there is a smaller disparity between the linear and non-linear critical values because the buoyancy provides a mechanism for transport of disturbance energy normal to the flow even in the linearized case.

APPLICATION OF THE METHOD TO THE CONSTANT STRESS LAYER; THE ASYMPTOTIC SOLUTION

We are going to apply this method to the constant stress wall layer of a turbulent boundary layer or channel flow with zero pressure gradient. Rather than attempt to find the general type of disturbance which is most efficient, we will use the result of Serrin (1959) and Joseph (1966) for the Couette flow, presuming that here also longitudinal rolls will be most efficient; thus we will assume a disturbance independent of x_1 (the streamwise direction) and periodic cross-stream. Setting $U_i = (U(x_2), 0, 0)$,

$$\begin{aligned} u_1 &= u e^{ikx_3} \\ u_2 &= ik\psi e^{ikx_3}, \quad u_3 = -\psi' e^{ikx_3} \end{aligned} \tag{18}$$

where u , and ψ are functions of x_2 , and a prime denotes differentiation with respect to x_2 , equation (9) becomes

$$\begin{aligned} \sigma^+ u + U' ik\psi/2 &= (vu'') - k^2 vu \\ ik\sigma^+ \psi + U' u/2 &= -\pi'/\rho + 2(vik\psi') - ikv(\psi'' + k^2\psi) \\ -\sigma^+ \psi' &= -\pi k/\rho - [v(\psi'' + k^2\psi)]' + 2vk^2\psi' \end{aligned} \tag{19}$$

Eliminating the pressure π between the second and third of (19) results in (writing $v^2 = D^2 - k^2$, $D = d/dy$)

$$\begin{aligned} \sigma^+ v^2 \downarrow + U' ik u / 2 &= (\nu v^2 \downarrow^2 + 2v' v^2 D + v'' [D^2 + k^2]) \downarrow \\ \sigma^+ u + U' ik \downarrow / 2 &= (\nu v^2 \downarrow + v' D) u \end{aligned} \quad (20)$$

In the constant stress layer of a turbulent boundary layer we may write $v = u_\tau z/U$, where u_τ is the shear velocity. Thus, in the viscous sublayer,

$$v = v_m \quad (21)$$

the molecular value, while in the logarithmic region we have

$$v \sim u_\tau K y \quad (22)$$

where K is von Karman's constant, ~ 0.4 . We may use Reichardt's (1951) expression, which agrees with measurement reasonable well through the entire region (see Figure 1):

$$v = v_m [1 + K (y^+ - R \tanh y^+ / R)] \quad (23)$$

where R is the point where the extended wall profile meets the extended logarithmic profile, roughly 11.6 and $y^+ = y_\tau / v_m$. We will find that our final answer is only weakly dependent on the particular form chosen for (23); it is only essential that the first two derivatives vanish at the wall. Using (23), it can be seen that v is zero in the sublayer, rises near R , and takes the constant value v_m in the logarithmic region; v' is zero in the sublayer and in the logarithmic region, being non-zero (and positive) only near $y^+ = R$. From the relationship $v = u_\tau z/U$, we can evidently express the mean velocity gradient also in terms of the function (23). If we non-dimensionalize y by the value corresponding to $y^+ = R$, say y_* so that the dimensionless $y = y^+ / R = y/y_*$, and with $\lambda = y_*^2 \sigma^+ / v_m$, then the equations can be written as

$$\begin{aligned} \lambda v^2 \downarrow + (R^2 ik / 2f) u &= f v^2 \downarrow^2 + 2v' v^2 D \downarrow + f'' (D^2 + k^2) \downarrow \\ \lambda u + (R^2 ik / 2f) \downarrow &= f v^2 u + f' Du \end{aligned} \quad (24)$$

with Reichardt's form for f

$$f = 1 + K R (y - \tanh y) \quad (25)$$

We see (c.f. Figure 2) that the term in R^2 is, for $R = 11.6$ (the true value in the boundary layer) nearly 10^2 times the terms on the right-hand side near $y = 1$. As y increases, the terms do not become comparable until $y \sim 10^1$. This suggests an asymptotic solution (Cole, 1968) using R^2 as the large parameter (but holding the implicit R which appears in the definition of f fixed*). In order to make a sensible (non-trivial) problem, we must assume that $\lambda = 0(R^2)$ also; that is, that outside the (viscous) wall layer production and growth balance, while in the wall layer, viscous loss is important.

*It appears to be possible to construct an expansion allowing this R to vary also; it is more complicated, however, in that three distinguished limits are found, and the behavior of the solutions in our range of parameters is the same.

Sufficiently far outside (for $y > 10^1$) there must again be a region where viscosity is important, but there the solution is already so small that we may safely ignore the transition to this new region^{**}. Thus, writing $\beta^2 = \lambda/R^2$, $R = \epsilon^{-1}$, we have

$$\begin{aligned} \beta^2 \nabla^2 \psi + (ik/2f) u &= \epsilon^2 (f \nabla^2 \psi + 2f' \nabla^2 D \psi + f'' (D^2 + k^2) \psi) \\ \beta^2 u + (ik/2f) \psi &= \epsilon^2 (f \nabla^2 u + f' u') \end{aligned} \quad (26)$$

Now, for the inner problem (next to the wall) we can write $\eta = y/\epsilon$, and

$$\begin{aligned} \psi &= v_0(\epsilon) (g_0(\eta) + \epsilon^2 g_1(\eta) + \dots) \\ u &= v_0(\epsilon) (h_0(\eta) + \epsilon^2 h_1(\eta) + \dots) \end{aligned} \quad (27)$$

The order of the boundary layer thickness was determined from the necessity of keeping the highest order derivative in the equations. The order of the second order terms in (27) was determined from the desire to obtain a g_1 , h_1 different from g_0 , h_0 . If necessary for the matching (it will not be) another g_0 and h_0 of order between ϵ^2 and 1 might be added in (27). The equality of the orders of the leading terms arises from the need to retain ψ terms in the u equation, and u terms in the ψ equation. The equations are (after cross-substituting)

$$\begin{aligned} g_0''' - \beta^2 g_0'' &= 0; g_1''' - \beta^2 g_1'' = ikh_0/2 + 2k^2 g_0'' - \beta^2 k^2 g_0 \\ h_0'' - \beta^2 h_0 &= ikg_0/2; h_1'' - \beta^2 h_1 = k^2 h_0 + ikg_1/2 \end{aligned} \quad (28)$$

The boundary conditions at the wall, of course, are $\psi(0) = \psi'(0) = 0$; $u(0) = 0$. Applying these, the solutions which we need become

$$\begin{aligned} g_0 &= D [1 - \beta\eta - e^{-\beta\eta}] \\ h_0 &= D [e^{-\beta\eta} - 1 + \beta\eta + (\beta\eta/2)e^{-\beta\eta}] ik/2\beta^2 \\ g_1 &= B(e^{-\beta\eta} - 1) + \beta\eta [B + (9/4 + 4\beta^4)k^2 D/24\beta^6] - (1-4\beta^4)(\beta^2\eta^2 - \beta^3\eta^3/3)k^2 D/8\beta^6 \\ &\quad - \beta\eta e^{-\beta\eta}(9/4 + 4\beta^4)k^2 D/24\beta^6 + \beta^2\eta^2 e^{-\beta\eta} k^2 D/32\beta^6 \end{aligned} \quad (29)$$

We have also discarded terms that are transcendentally large as $\eta \rightarrow \infty$, which would preclude matching. The constant B must be found in terms of D by the matching. D , of course, must remain arbitrary, since the equations are homogeneous. In addition, of course, we will have to determine v_0 from the matching.

It is interesting to note in equation (28) that to zeroth order ψ (i.e. - the lateral motion) simply experiences a balance between decay and dissipation; the streamwise motion, however, experiences some production due to the lateral motion.

^{**} That is, we will find that it is the matching to the solution between 1 and 10^1 that determines the eigenvalue relations to the order in which we are interested.

This agrees with the conclusions of Joseph (1971). From the equation for h_0 it is clear that the vortices are sweeping slow moving fluid up from the wall (and vice versa) producing disturbances in the streamwise motion.

We may now attack the outer problem. Since the amplitude of the solution is arbitrary, we may take it to be of order unity, writing

$$\begin{aligned}\psi &= G_0(y) + \epsilon^2 G_1(y) + \dots \\ u &= H_0(y) + \epsilon^2 H_1(y) + \dots\end{aligned}\tag{30}$$

where, again, the order of the second term is fixed by requiring that G_1 be different from G_0 (a term of intermediate order similar to G_0 could be included, but will not be necessary). To first order we have

$$\begin{aligned}\beta^2 H_0 + (ik/2f) G_0 &= 0 \\ \beta^2 v^2 G_0 + (ik/2f) H_0 &= 0\end{aligned}\tag{31}$$

The first of these is the "mixing length" assumption, frequently made (see Bakeswell, Lumley, 1967): that the perturbations in u are those produced from the mean velocity gradient by the lateral motion. Combining the two equations (31), we obtain

$$G_0'' - k^2 (1 - 1/4f^2\beta^4) G_0 = 0\tag{32}$$

It is fairly simple to show that this solution cannot match with (29) unless there is a turning point - i.e. - unless $1 - 1/4f^2\beta^4$ has a zero somewhere. This is a simple kinematic requirement: if (32) has no turning point, the lateral velocity (outside the viscous layer) is monotone with distance from the surface, producing upward and downward streaming jets, rather than cells. These do not satisfy the requirement that the stream function and its derivative must vanish at infinity. If (32) has one turning point, we have one cell (see Figure 3). Two turning points correspond to two cells, each on top of the other, and so on. Clearly, more cells means a smaller length scale, and hence greater dissipation. Thus, the cases of two and more turning points correspond to a lower growth rate. We will consider only the case of a single turning point.

A simple form of the solution for (32) may be obtained (Cole, 1968) formally valid for large k (but probably satisfactory for k as small as 3) by the two variable expansion procedure. If we designate by y_c the point where

$$2\beta^2 f(y_c) = 1\tag{33}$$

then for $y > y_c$ we can write

$$G_0 = [1 - 1/4f^2\beta^4]^{-1/4} e^{-ky} +, \quad Y_+ = y \int_c^y [1 - 1/4f^2\beta^4]^{1/2} dy\tag{34}$$

where we have discarded the positive exponential. Now, for $y < y_c$, the solution takes the form of trigonometric functions. The proper branch (i.e. - the continuation of (34) may be selected by constructing an inner expansion of (32) near the turning point, as is discussed in Cole (1968) (see Figure 4). The result is for $y < y_c$:

$$G_0 = 2\sqrt{2/3} (\sin \pi/12 + \cos \pi/12) [1/4\beta^4 f^2 - 1]^{-1/4} \sin(3\pi/4 - kY_-)$$

$$Y_- = \int_{y_c}^{y_c} [1/4\beta^4 f^2 - 1]^{1/2} dy$$
(35)

For smaller k , the coefficient in (12) becomes a series in k^{-1} , the terms involving integrals and derivatives of $1/4\beta^4 f^2 - 1$. A little manipulation of the equations is sufficient to show, however, that, as written in (35), the coefficient has vanishing first and second derivatives at the wall, while the full series representation of the coefficient has vanishing first through third derivatives, due to the vanishing of the first two derivatives of f . (Substitution of a form such as (35) with arbitrary coefficient into (32) gives an equation for the coefficient, from which the derivatives at the wall may be determined; if the first derivative vanishes, the first three derivatives vanish. But a non-vanishing first derivative produces a term which could only be matched with (29) if f' or f'' had been assumed not to vanish at the wall.) Thus, although the proper form of the coefficient for finite k will change the shape of G_0 away from the wall somewhat, it will not affect the matching through third order; we will need only second order.

The second order equations for G_1 and H_1 are, of course,

$$\beta^2 v^2 G_1 + ikH_1/2f = fv^2 v^2 G_0 + 2f' v^2 DG_0 + f'' (D^2 + k^2) G_0$$
(36)

$$\beta^2 H_1 + ikG_1/2f = fv^2 H_0 + f' H_0'$$

and the right hand sides may be evaluated in terms of G_0 and G_0' using equations (31) and (32). We are primarily interested in the behavior of the solutions near the wall, however. If account is taken of the fact that $G_0(0)=0(\epsilon)$, $G_0'(0)=0(1)$ (which we can only know from the first order matching; i.e. that the inviscid boundary condition is nearly satisfied, while the no slip condition is not) then the leading term (through linear terms in y) is

$$\beta^2 v^2 G_1 + ikH_1/2 \approx (k^2/4\beta^4)^2 G_0$$
(37)

$$\beta^2 H_1 + ikG_1/2 \approx (ik^3/8\beta^6) G_0$$

or

$$G_1'' + k^2 (1/4\beta^4 - 1) G_1 \approx (2k^4/16\beta^{10}) G_0$$
(38)

To linear terms, then, the particular solution will be

$$(G_1)_p \approx (2k^2/16\beta^{10}) (1/4\beta^4 - 1)^{-1} G_0$$
(39)

This result is the mathematical expression of the physical truth that near the wall, the variation of viscosity with y is not important.

The general solution will be exactly like (35) - we can thus write to the order of linear terms in y ,

$$\psi = G_0 + \epsilon^2 [C G_0 + (2k^2/16\beta^{10}) (1/4\beta^4 - 1)^{-1} G_0] + \dots$$
(40)

where C is an unknown constant that must be determined from the matching.

We may now proceed to the matching, using an intermediate limit in which $y/\delta = y_\delta = 0$ (1), $\delta(\epsilon) \rightarrow 0$, $\delta/\epsilon \rightarrow \infty$, so that $\eta = \delta y_\delta / \epsilon \rightarrow \infty$. The inner solution becomes (we will do the matching only in ψ)

$$\begin{aligned}\psi &= v_0 D (1 - \beta \delta y_\delta / \epsilon) + \epsilon^2 v_0 \left\{ -B + \beta (\delta y_\delta / \epsilon) [B + (9/4 + 4\beta^4) k^2 D / 24\beta^6] \right. \\ &\quad \left. - (1 - 4\beta^4) [\beta^2 (\delta y_\delta / \epsilon)^2 - \beta^3 (\delta y_\delta / \epsilon)^3 / 3] k^2 D / 8\beta^6 \right\} + \dots\end{aligned}\quad (41)$$

neglecting transcendentally small terms.

Writing

$$3\pi/4 - kY_+ = \mu(\epsilon) + k(1/4\beta^4 - 1)^{1/2} \delta y_\delta + O(\delta^4) \quad (42)$$

near the wall, we can write (to third order)

$$G_0 = P \sin [\mu(\epsilon) + k(1/4\beta^4 - 1)^{1/2} \delta y_\delta + \dots] \quad (43)$$

where P is a constant. Thus, through terms $O(\delta^3)$, $O(\mu^3)$,

$$\begin{aligned}G_0 &= P \left[\mu - \mu^3/6 + \delta y_\delta k (1/4\beta^4 - 1)^{1/2} (1 - \mu^2/2) - k^2 (1/4\beta^4 - 1) (\mu - \mu^3/6) (\delta y_\delta)^2/2 \right. \\ &\quad \left. - k^3 (1/4\beta^4 - 1)^{3/2} (1 - \mu^2/2) (\delta y_\delta)^3/6 + \dots \right]\end{aligned}\quad (44)$$

and, from (40), ψ is given by

$$\begin{aligned}\psi &= (44) + Pe^2 \left[C + (2k^2/16\beta^{10}) (1/4\beta^4 - 1)^{-1} \right] \\ &\quad \left[\mu - \mu^3/6 + \delta y_\delta k (1/4\beta^4 - 1)^{1/2} (1 - \mu^2/2) + \dots \right]\end{aligned}\quad (45)$$

where the second term is correct to $O(\delta)$.

Matching the term of $O(\delta)$ in (45) we find that we must have $v_0 = \epsilon$,

$$-\beta D = Pk(1/4\beta^4 - 1)^{1/2} \quad (46)$$

Writing

$$\mu(\epsilon) = w_1 \epsilon + w_2 \epsilon^2 + w_3 \epsilon^3 + \dots \quad (47)$$

we may match the term of $O(\epsilon)$ in (41), giving

$$D = Pw_1 \quad (48)$$

Since there is no term of $O(\epsilon^2)$ in (41), $w_2 = 0$. Matching terms of $O(\epsilon^3)$, we have

$$-B = Pw_3 - w_1^3 P/6 + P[C + (2k^2/16\beta^{10}) (1/4\beta^4 - 1)^{-1}] w_1 \quad (49)$$

Terms of $O(\epsilon^2 \delta)$ give

$$\begin{aligned} \beta [B + (9/4 + 4\beta^4)k^2 D/24\beta^6] &= - Pk(1/4\beta^4 - 1)^{1/2} w_1^2/2 \\ &+ P[C + (2k^2/16\beta^{10})(1/4\beta^4 - 1)^{-1}]k(1/4\beta^4 - 1)^{1/2} \end{aligned} \quad (50)$$

Terms of $O(\epsilon \delta^2)$ give

$$-(1 - 4\beta^4)\beta^2 k^2 D/8\beta^6 = - Pk^2(1/4\beta^4 - 1)w_1/2 \quad (51)$$

and of $O(\delta^3)$

$$+(1 - 4\beta^4)\beta^3 k^2 D/24\beta^6 = - Pk^3(1/4\beta^4 - 1)^{3/2}/6 \quad (52)$$

As a result of (46), (51) and (52) are satisfied exactly. Equations (49) and (50) represent two equations in three unknowns; fortunately two of the unknowns occur in the same combination, giving

$$w_3 = -(1/4\beta^4 - 1)^{1/2}(1/48\beta^4 + 1)k^3/2\beta^3 \quad (53)$$

The other combination of constants need not concern us.

We can thus write, using (46), (47), (48) and (53), our eigenvalue conditions:

$$3\pi/4 - kY_{-0} = -(1/4\beta^4 - 1)^{1/2}\epsilon k/\beta - (1/4\beta^4 - 1)^{1/2}(1/48\beta^4 + 1)\epsilon^3 k^3/2\beta^3 + \dots \quad (54)$$

to $O(\epsilon^5)$, where Y_{-0} is given by

$$Y_{-0} = \int_0^{y_c} [1/4\beta^4 t^2 - 1]^{1/2} dy \quad (55)$$

Expression (54) gives the offset (see Figure 5) by which the outer (inviscid) solution fails to meet the inviscid boundary condition to leave room $O(\epsilon)$ for the viscous (eddy) boundary layer. Equation (54) has, in general, two positive roots for k (for fixed ϵ, β); we are interested in the critical condition when there is only one (double) root. This may be obtained by differentiating the expression (54) with respect to k , requiring that the derivative vanish also at the same value of k . If we indicate by k_+ this critical value, we obtain (noting that $\beta/\epsilon = \sqrt{\lambda}$)

$$\begin{aligned} (\sqrt{\lambda}/k_+)^3 &= (4/3\pi)(1/4\beta^4 - 1)^{1/2}(1/48\beta^4 + 1) \\ Y_{-0} \sqrt{\lambda} &= (1/4\beta^4 - 1)^{1/2} + (1/4\beta^4 - 1)^{1/6}(1/48\beta^4 + 1)^{1/3}(3/2^{1/3})(3\pi/8)^{2/3} \end{aligned} \quad (56)$$

If now a value of β is selected, Y_{-0} may be calculated, as well as $k_+/\sqrt{\lambda}$ and $\sqrt{\lambda}/Y_{-0}$; from the latter then we have $\sqrt{\lambda}$, and from the former, k_+ , while from $R = \sqrt{\lambda}/\beta$ we obtain R .

If Figure 6 we show a plot of λ and k_+ versus R (holding the R in f fixed at 11.6). It is worth noting that the form of the viscosity outside the sublayer did not influence the solution; the only property we used was the vanishing of the first two derivatives at the wall, a kinematic requirement in a zero pressure gradient. The form of the mean shear only influences expression (55).

COUETTE FLOW

The amount of algebra involved in arriving at (56) is such as to make one suspicious of the result in the absence of an independent check. Fortunately, one is available. The case of Couette flow, solved by Serrin (1959) and Joseph (1965) can be handled by the same technique. The inner solution (29) remains the same, as does the outer solution; we have only to set $f \equiv 1$; the fact that f varied affected the matching only through the value of Y_{-0} . To determine the value of Y_{-0} , we have only to determine the value of y_c , and here we must replace our previous condition (that ψ decay exponentially) by a requirement that ψ be symmetric about the mid-point of the channel. Thus y_c must be fixed so that $3\pi/4 - k Y_{-0} \Big|_{1/2} = \pi/2$, or

$$3\pi/4 - k(1/4\beta^4 - 1)^{1/2} (y_c - 1) = \pi/2 \quad (57)$$

Substituting in (54), we have

$$\pi/2 - k(1/4\beta^4 - 1)^{1/2} = -(1/4\beta^4 - 1)^{1/2} \epsilon k/\beta - (1/4\beta^4 - 1)^{1/2} (1/48\beta^4 + 1) \epsilon^3 k^3 / 2\beta^3 + \dots \quad (58)$$

We may thus go directly to (56), substituting $\pi/2$ for $3\pi/4$, and $(1/4\beta^4 - 1)^{1/2}$ for Y_{-0} ; thus, we have

$$\begin{aligned} (\sqrt{\lambda}/k_+)^3 &= (2/\pi)(1/4\beta^4 - 1)^{1/2} (1/48\beta^4 + 1) \\ (1/4\beta^4 - 1)^{1/2} \sqrt{\lambda} &= (1/4\beta^4 - 1)^{1/2} \\ + (1/4\beta^4 - 1)^{1/6} (1/48\beta^4 + 1)^{1/3} (3/2)^{1/3} (\pi/4)^{2/3} \end{aligned}$$

The same computations have been carried out using these equations; the results are also shown in Figure 6. It may be seen that the curve of λ vs. R extrapolates through the exact value of $R = 4.54$ as $\lambda \rightarrow 0$, while the curve of k_+ extrapolates to the exact value of 1.56 at the same value of R .

We may conclude from this that our estimate for λ is satisfactory for $\lambda \geq 20$, while the k estimate is somewhat less reliable, being good for perhaps $k \geq 5$. The bending away of both curves below these points must be attributed to progressive failure of the approximation.

DISCUSSION OF THE RESULTS

Examining Figure 6, we evidently have at $R = 11.6$ a growth rate of $\lambda \sim 31.5$ and a $k_+ \sim 3.7$ (the latter value being somewhat less reliable). The critical Reynolds number for this type of flow (corresponding to $\lambda = 0$) is evidently about 9.2. That this should be larger than that for the Couette flow can be justified by noting that in the Couette flow the top halves of the eddies are subjected to the same strain rate, and hence are self-sustaining; in the wall-layer flow, the shear drops rapidly, and the outer parts of the eddies must be sustained by the inner parts; hence the shear must be proportionately higher, or the dissipation lower, resulting in a higher critical Reynolds number.

The growth rates are quite large measured in sublayer variables; since sublayer time scales are themselves the shortest in a wall flow, the evolution of these eddies must be regarded as extremely rapid. In this connection, the observations of Kline and his group (e.g. - Kline, et al., 1967) are relevant; wall eddies of this general form were observed to "burst," i.e. - evolve very rapidly relative to other sublayer phenomena.

In other respects the dynamics of the eddies (from the asymptotic analysis) agree well with observations, in particular the "mixing length approximation" (eg. 31a), which has often been used (Townsend, 1956; Bakewell and Lumley, 1967; Payne and Lumley, 1967).

The single disappointing feature of these eddies is their size. Referred to sub-layer variables, $k_+ = 3.7$ corresponds to a transverse wave length of roughly 20, while observed wave lengths (e.g. Bakewell and Lumley, 1967) are closer to 100. Our eddies have their apparent centers (the maxima of ψ) at about 10, while observed eddies seem to have their centers closer to 30. Evidently these eddies are too small by a factor between 3 and 5; can this be explained?

In our considerations, we have not taken into account the effect of the presence of the eddy on the undisturbed velocity profile; to an inviscid first order approximation the mean shear will be reduced (outside the viscous region at the wall) by an amount $k^2 \psi' / 2g^2 (1/2g^2 \sim 2)$ when $R = 11.6$. This reduction in shear will favor the growth of an eddy with the same value of k_+ , but a smaller k due to the greater thickness of the layer. That is, as the shear is reduced, all length scales will grow to keep R, k_+ roughly constant. Hence, the k we have found is evidently a starting value; evolution of the eddy will involve progressive decrease in the dimensional k , until the disturbance energy reaches its peak and begins to decay. Viewed another way, as the shear is reduced, the effective Reynolds number is being reduced* - the eddy still feels the same effect of viscosity at the wall, but has a smaller energy source; we may expect the eddy to move down the curve toward $\lambda = 0$, having at that point a $k_+ \sim 0.7$ (from rather uncertain extrapolation of the curve of Figure 6). This corresponds roughly to the value observed, in sublayer variables, of ~ 100 .

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This work grew out of a numerical analysis carried out by Elswick (1967) for the same flow with $\sigma = 0$; although the results of that analysis now appear to be somewhat questionable, and have not been used here, its seminal role in this investigation must be acknowledged.

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* Note that the form of the velocity profile only affects ψ_0 ; a reduction in shear reduces ψ_0 but keeps the relation (54) otherwise unchanged.

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APPENDIX I

Excerpt from: G. I. Petrov, On the Growth of Oscillations in a Viscous Liquid and Transition to Turbulence. Publication Number 345, Transactions of the Central Aero-Hydrodynamic Institute, Moscow, 1938: pp. 20-24.

Lorentz, selecting an initial velocity distribution in the form of elliptical vortices, obtained:

$$R = U_{\max} h/v = 72$$

From a more precise statement of the problem, Karman and Orr obtained an even lower value $R \sim 42$, which obviously does not correspond to reality.

Below, we will attempt to give an explanation of this fact, and to show how this problem may be correctly formulated.

The linearized equation for the disturbance stream function, superimposed on a stationary plane parallel flow, has the form:

$$\Delta\Delta\psi - R[U \partial\Delta\psi/\partial x - U'' \partial\psi/\partial x + \partial\Delta\psi/\partial t] = 0. \quad (8)$$

As we have already shown, it is convenient to assume a solution of the form $f(x, y)e^{-i\beta t}$, i.e., in the form of an oscillation, the amplitude of which is a function of x and y . The distribution of the initial amplitude, or the function f , may be obtained from the equation:

$$\Delta\Delta f - R[U \partial\Delta f/\partial x - U'' \partial f/\partial x - i\beta\Delta f] = 0 \quad (9)$$

and corresponding boundary conditions. This equation is of the fourth order, and cannot be placed in self-adjoint form. The coefficients of the equation contain two characterizing parameters R and β , of which the first, in accordance with its physical interpretation, is always real, while the second is in general complex. Thus, the problem may be stated as: for a given value of R (which defines the basic flow) find a system of eigenvalues β and a corresponding system of functions $f(x, y)$, assuming that an arbitrary initial disturbance may be expanded in a series of these functions, or, in a different formulation, find a value of R which is associated in some way with a definite region of the values of β (for example, the minimum value of R , below which all given oscillations will decay, i.e., the imaginary part of β will be negative). This value of R is ordinarily used in calculating critical numbers, both as we have seen, in connection with the problem of stability of the flow of a viscous fluid and, as it is ordinarily formulated, the determination of the point of transition of a laminar flow to turbulence.

Neither of these formulations is, at first glance, obvious. The first, because it is not known with certainty whether the complete solution, expanded in a series of periodic oscillations of the form $f(x, y) e^{-i\beta t}$, will decay. And the second because the critical number R determines an initial growth diverging from the real velocity of the fluid, and this change in the regime determines a change in the mean velocity field.

Let us look at several general properties of our system. As we have already remarked, equation (9) cannot be placed in a self-adjoint form, and the eigenvalues of our system will in general be complex, in addition to the obvious difference from systems ordinarily considered in the theory of oscillations, that the characteristic parameters appear in the coefficients of the derivatives.

Let us examine the following expression:

$$\iint \bar{f}(x, y) L(f(x, y)) dx dy$$

the integral being taken over the region of the flow.

Here:

$$L(f) = \Delta f - R(U \partial f / \partial x - U'' \partial^2 f / \partial x^2 - i\beta \Delta f),$$

i. e., the right-hand (sic) side of equation (9).

This expression can be transformed to the following form:

$$\begin{aligned} \iint \bar{f}(x, y) L(f(x, y)) dx dy &= \iint \Delta f \Delta f dx dy + R \iint (U \partial f / \partial x \Delta f + \\ &+ U'' \bar{f} \partial f / \partial x + i\beta \bar{f} \Delta f) dx dy + \dots \end{aligned} \quad (20)$$

plus a contour integral.

If the function $f(x, y)$ satisfies equation (9), then expression (20) must equal zero, and we may determine R as the ratio of the two integral expressions, and seek the critical value as the minimum of this ratio, setting β_i equal to several values.

Since we have given boundary conditions, let us examine the variation of just the double integrals. The problem of the determination of the critical value may be set as an isoperimetric problem, i. e., to find the extremum of the functional $I_1 = \iint \Delta f \Delta f dx dy$ under the condition

$$I_2 = \iint - [U \partial \bar{f} / \partial x \Delta f + U'' \bar{f} \partial f / \partial x + i\beta \bar{f} \Delta f] dx dy = 1.$$

The Euler equation for our problem will be equation (9), and for \bar{f} the equation of its adjoint:

$$\Delta \bar{f} - \lambda [U \partial \Delta \bar{f} / \partial x + 2U' \partial^2 \bar{f} / \partial x \partial y - i\beta \Delta \bar{f}] = 0. \quad (21)$$

But, in order that our integrals have the same physical interpretation as in the work of Lorentz, Karman, et. al. which we described above, we must assume that \bar{f} is the complex conjugate of a complex function f , i. e., that

$$f = f_r + i f_i, \text{ to } \bar{f} = f_r - i f_i.$$

All of our conclusions follow from the form of the initial disturbance, i. e., by taking into account that the coefficients of our equation do not depend on x . taking $f = \phi e^{i\alpha x}$ and then $f = \bar{\phi}(y) e^{-i\alpha x}$ (where $\bar{\phi}(y)$ is the complex conjugate of $\phi(y)$).

Then the expression (2) takes the following form:

$$\begin{aligned} &0 \int^1 (\bar{\phi}'' \phi'' + 2\alpha^2 \bar{\phi}' \phi' + \alpha^4 \bar{\phi} \phi) dy + R \int [\alpha U 1/2 (\bar{\phi} \phi' - \phi \bar{\phi}') - \beta_i (\phi' \bar{\phi}' + \alpha \bar{\phi} \phi)] dy \\ &- i\alpha R \int [(U - c_r) (\bar{\phi}' \phi' + \alpha^2 \bar{\phi} \phi) + 1/2 U'' \bar{\phi} \phi] dy \end{aligned} \quad (20a)$$

The integrals are taken over a region contained between the walls, and over a wavelength of the given disturbance, so that the contour integral will vanish.

The real part of expression (20a) can be regarded as the change in energy in a strip of one wavelength of the disturbance, and may be obtained by examining the expression for $dE/dt dx dy$ (see expression (6)), taking into account that the real part of the stream function is $\psi = \phi(y)e^{i(\alpha x - \beta t)}$ and integrating in x over one wave length.

The same functional can be obtained as an expression $\int_0^1 (\bar{\phi} L(\phi) + \phi \bar{L}(\bar{\phi})) dy$, where $L(\phi)$ is the left-hand side of equation (13), while $\bar{L}(\bar{\phi})$ is the corresponding expression for the conjugate function, and has the form:

$$\bar{L}(\bar{\phi}) = \frac{-IV}{\bar{\phi}} - 2a^2 \bar{\phi}'' + a^4 \bar{\phi} + iaR[(U - \bar{c})(\bar{\phi}'' - a^2 \bar{\phi}) - U'' \bar{\phi}],$$

i.e., - the left-hand side of equation (13) with the complex parameters changed to their conjugates.

The imaginary part of expression (2) may be obtained as an expression

$$\int_0^1 (\bar{\phi} L(\phi) - \phi \bar{L}(\bar{\phi})) dy$$

Both of these expressions for functions, satisfying equation (13), must vanish. Then from the real part we may express R , or β for given R , as a ratio of functionals. The imaginary part satisfies a Zol'berg condition, from which it follows that for undisturbed velocity profiles not having an inflection point, c_r is always less than U_{max} , and we may obtain the phase velocity c_r or the wave number S_r as a ratio of functionals.

But in order that the disturbance be dynamically possible, i.e., satisfy equation (13), the real and imaginary parts must vanish simultaneously, and the problem of the critical value must be formulated as a problem of the extremum of $\int_0^1 (\bar{\phi}' \phi'' + 2a^2 \bar{\phi}' \phi' + a^4 \bar{\phi} \phi) dy$ under the condition

$$\int_0^1 [1/2 aU(\bar{\phi}' \phi - \bar{\phi} \phi') + ia[(U - c_r)(\bar{\phi}' \phi' + a^2 \bar{\phi} \phi) + 1/2 U \phi' \bar{\phi}]] dy = 0$$

(which includes the condition that the imaginary part vanish). The Euler equations for this problem are equation (13) for ϕ and its adjoint for $\bar{\phi}$, i.e.,

$$\frac{-IV}{\phi} - 2a^2 \phi'' + a^4 \phi - iaR[(U - c)(\bar{\phi}'' - a^2 \bar{\phi}) + 2U' \bar{\phi}'] = 0.$$

But, if ϕ satisfies equation (13), then the function must satisfy the equation $\bar{L}(\bar{\phi}) = 0$. This is possible only if the following relation holds:

$$(U - c_r)(\bar{\phi}'' - a^2 \bar{\phi}) - 1/2 U'' \bar{\phi} + U' \bar{\phi}' = 0, \quad (22)$$

i.e., - if equation (13) and its conjugate are anti-symmetric.

But the condition (22) very much reduces the class of functions which can give our solution. Since the solution of equation (22) depends only on two arbitrary parameters, the solution of the variational problem in such a formulation is apparently impossible. This explains the failure of all the attempts to resolve the problem of the so-called energy method, since apparently disturbances obtained from the minimum value of the ratio which determines R , are dynamically impossible.

Thus, the critical number R may be found as the smallest value of the ratio $I_1/R(I_2)$ (where $R(I_2)$ is the real part of the complex quantity I_2), in which are substituted solutions of equation (9), but these functions do not give an extremum

of this ratio. It may also be said regarding the eigenvalues $\beta = \beta_r + i\beta_i$, that in the system under consideration they apparently do not possess an extremal character.

Translator's Note: Equation (13) is:

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi - i\alpha R[(\bar{U} - c)(\phi'' - \alpha^2\phi) - U''\phi] = 0$$

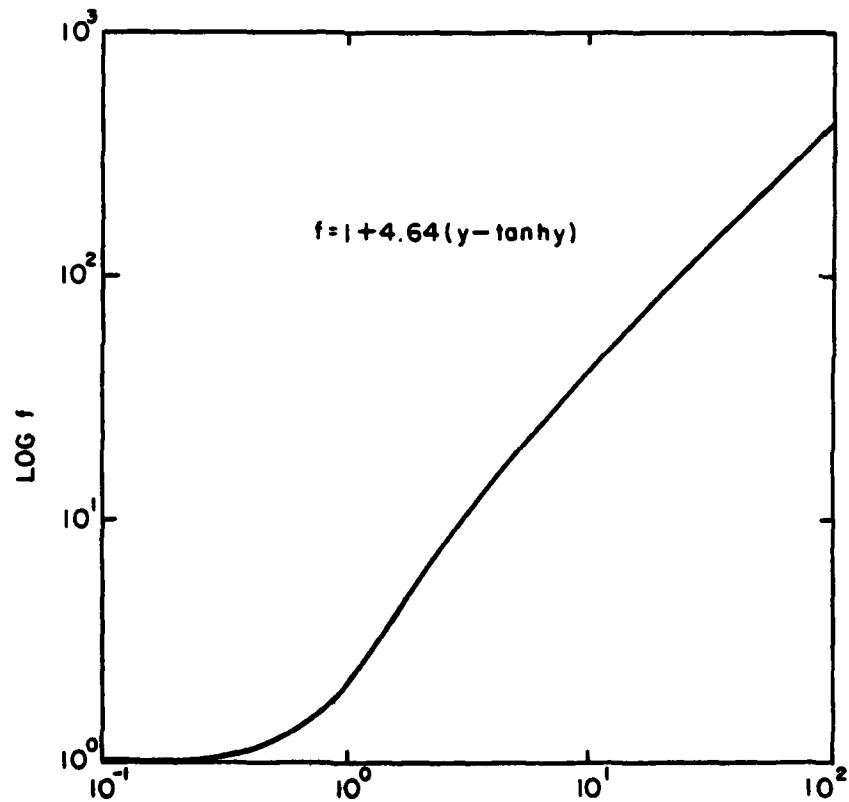


Figure 1 Reichardt's (1951) Viscosity Expression.

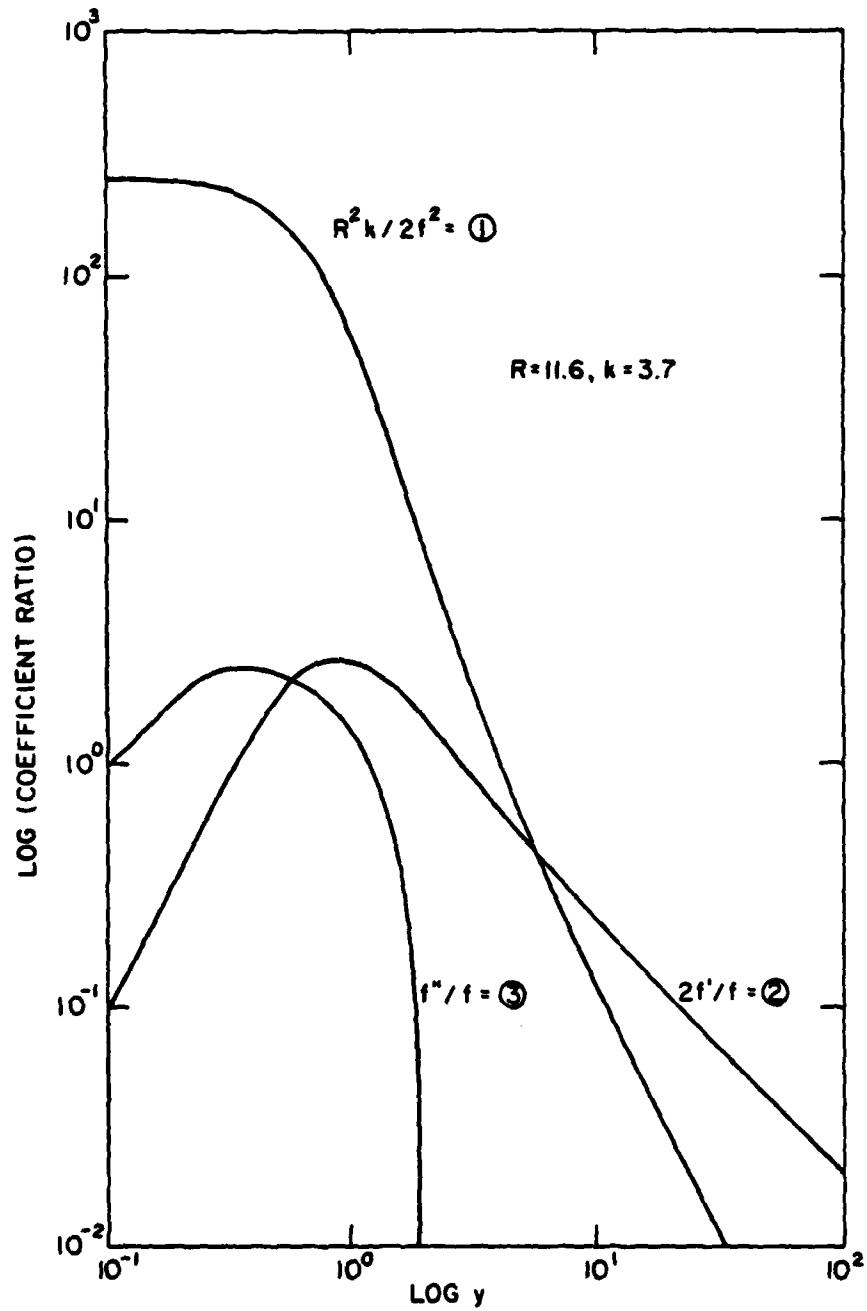
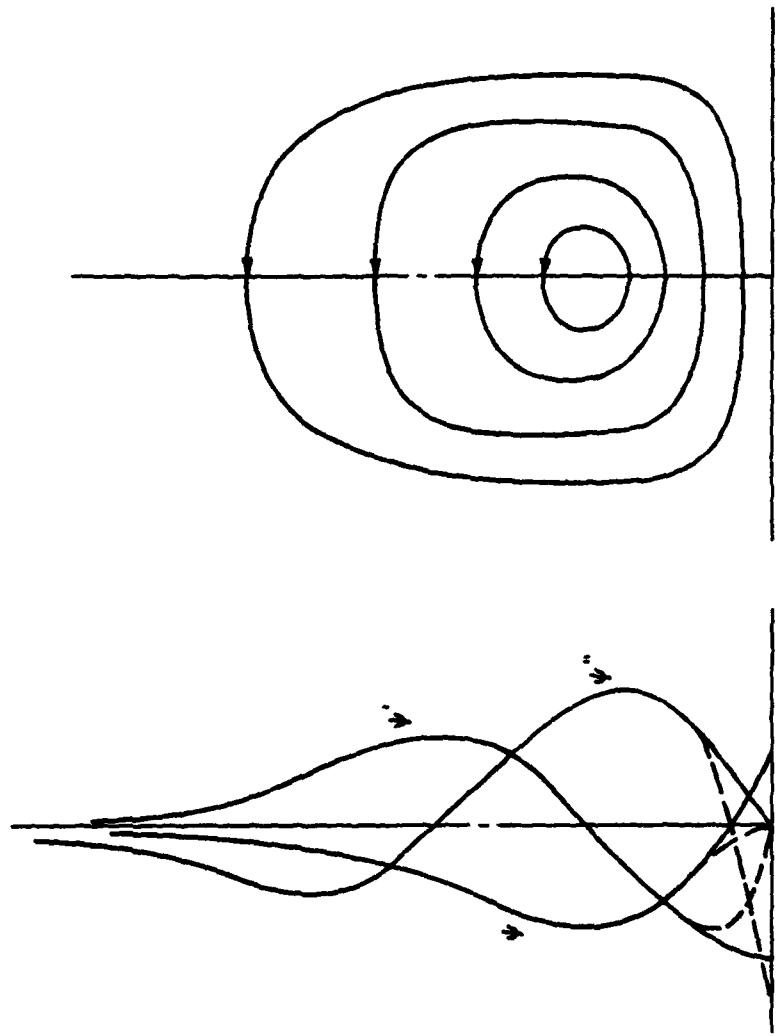


Figure 2, Ratio of the Coefficients of the Production Term 1,
the f' Term 2, and the f'' Term 3 to f .

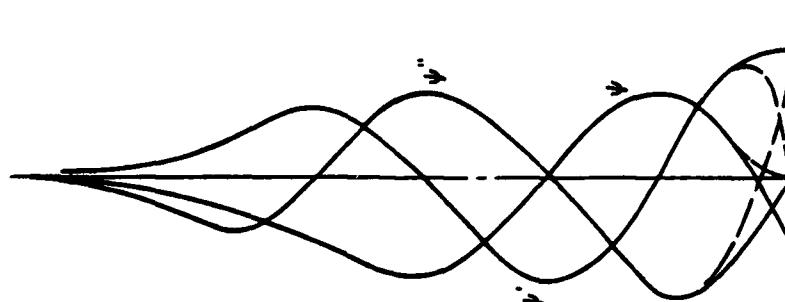
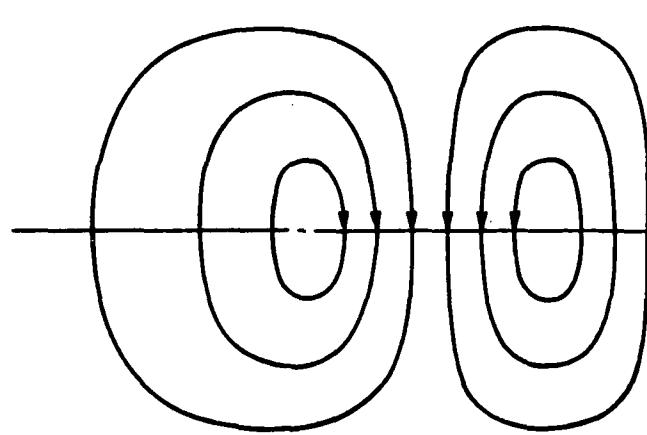


(a) ONE TURNING POINT

Figure 3 Structure of ψ and Corresponding Cells for One and Two Turning Points. Note Change due to Viscosity
(dashed curves).

(b) TWO TURNING POINTS

Figure 3 Continued



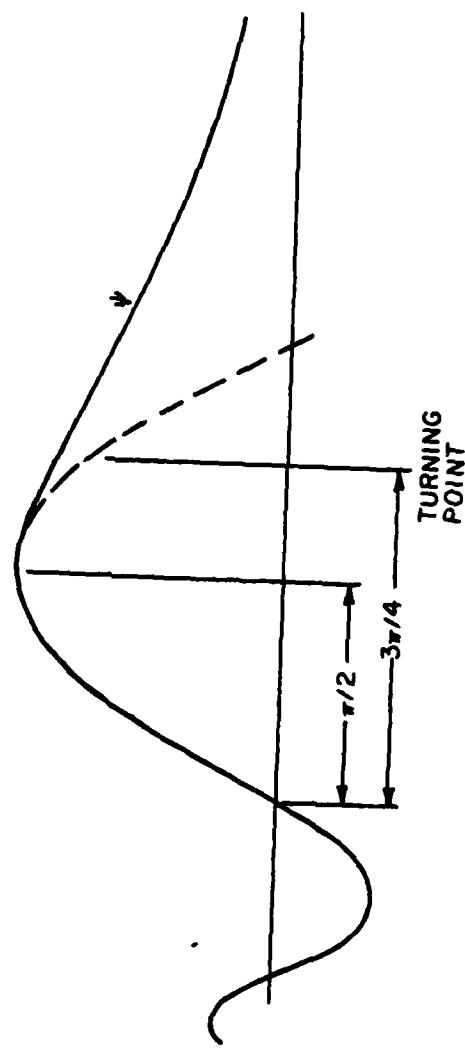


Figure 4 Form of the First Order Outer Solution, After Cole (1968). The Corresponding Solution for Couette Flow is Sketched (dashed).

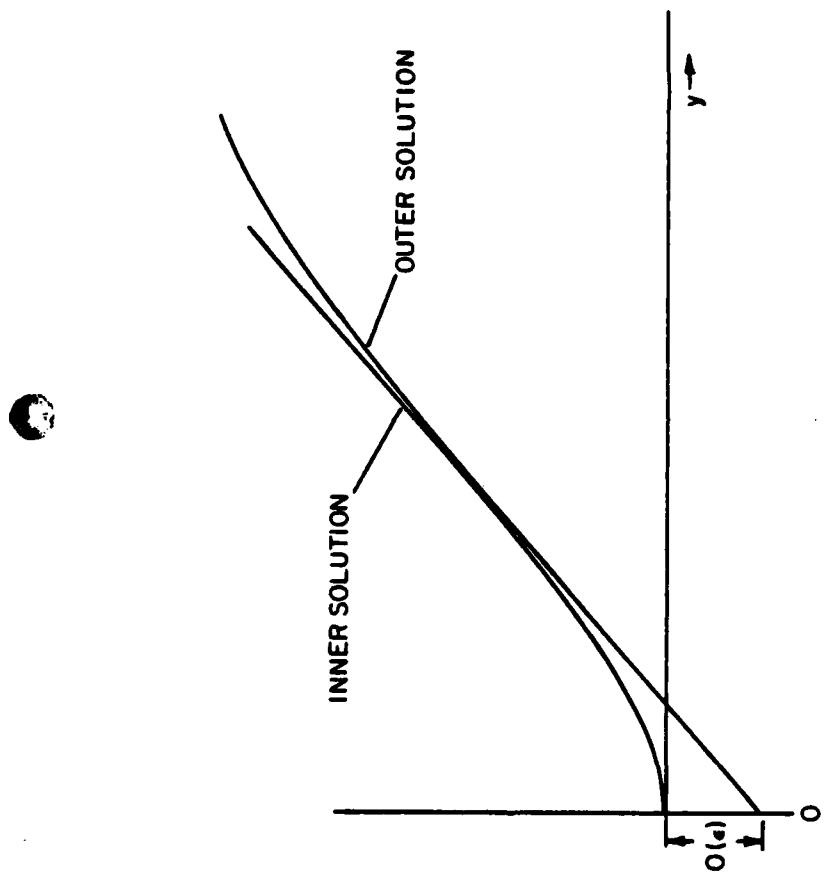


Figure 5 Illustration of the Offset Required for Matching.

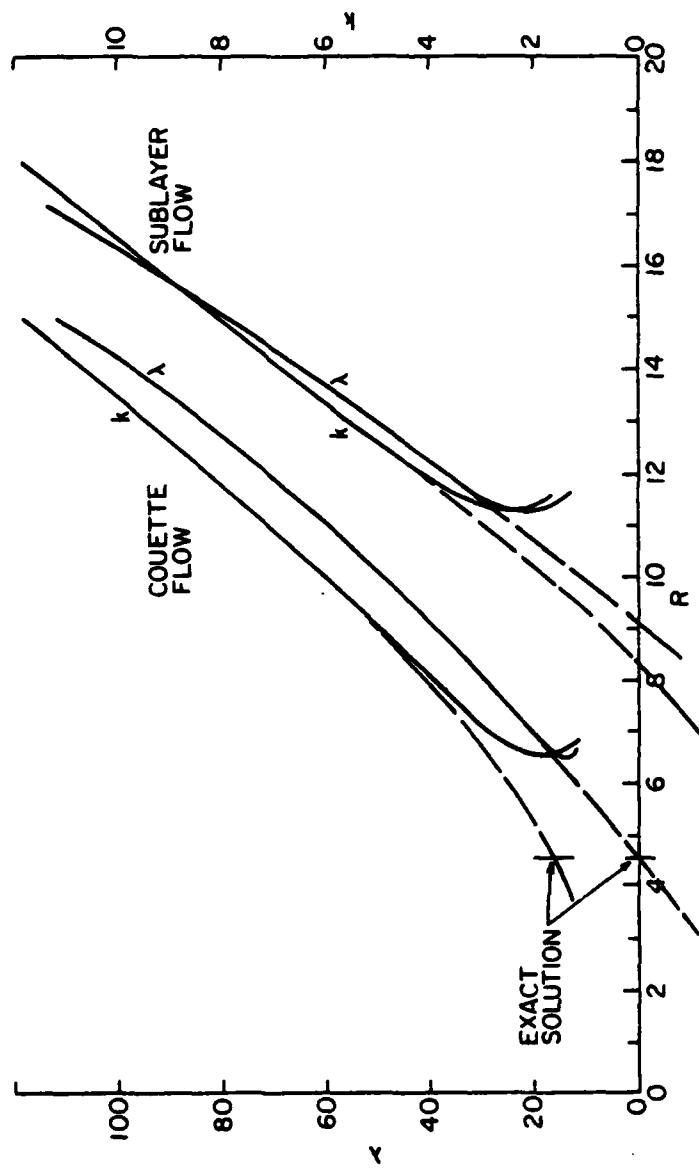


Figure 6 Plot of Growth Rate and Wave Number vs. Reynolds Number for Couette Flow and for the Boundary Layer.

